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THE CAUCHY PROBLEM FOR AN ELASTIC DIELECTRIC IN A MAGNETIC FIELD

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1. Introduction

In Ref. [1], the problem of magnetoelasticity is treated for a body with good conductivity, for which the conduction currents dominate the displacement currents. It was found to be possible to reduce the resolving functions (described in the general case of coupled mechanical-electromagnetic fields by partial differential equations of the 12th order) to a simplified form, such that they are described by equations of the 2nd and 4th order, which facilitates in an essential manner the solution of this complicated problem. In addition the solutions were constructed so as to make use of the known equations of the classical theory of elastic and anelastic bodies.

In Ref. [2], an analogous problem was solved in quadratures for the much simpler problem of a perfect conductor, for which the equations describing the resolving functions were also of the 2nd and 4th order. The object of the present paper is to obtain a solution of the Cauchy problem for the other extreme case (the first being that of [1]), that of a dielectric, where the conduction currents are zero. Considering mechanical-electromagnetic fields excited by mechanical agents, simplified solutions are obtained in this case also, similarly to [1]. The general 12th order partial differential equations for the resolving functions are reduced to ordinary wave equations and a fourth order equation, the latter containing, in a particular case, a small parameter and reducing to a Volterra integro-differential equation with a kernel constituting a combination of solutions of wave equations. The equations and solutions for dielectrics are much simpler than for conductors, [1].

In the solution of the general problem there is an unsolved case, where the conduction currents are of the order of the displacement currents. In this case, it is not possible to find such simple solutions as those of [1] and the present papers, the full solution of the problem thus remaining an open question. The idea of the construction of the resolving functions is the same in the present paper as in Ref. [1]. The notations are also so chosen that the system of equations is apparently identical with those solved in [1]. The similarity is apparent because the same symbols correspond to different operators and parameters. In spite of the identity of the methods for constructing the resolving functions with those used in [1], the basic argument will be repeated here in view of the two papers being written in

parallel, and because that argument is of essential meaning for the principal stage of the solution procedure, that is, the construction of the simplified equations which are of a different and much simpler form than those of [1].

In every case where the conclusions of [1] are preserved we use them without proof (for instance, in the case of the construction of the operational resolving functions for anelastic bodies).

2. The Resolving Functions

The linearized equations of a mechanical and electromagnetic field for an isotropic real conductor in an initially uniform magnetic field $\mathbf{H}$ have in the general case (that is where the conduction and displacement currents are of the same order) the form:

$$
\tag{2.1}
\begin{cases}
\operatorname{rot} \mathbf{h} = \dfrac{4\pi}{c}\mathrm{j} + \dfrac{\varepsilon}{c}\dfrac{\partial \mathbf{E}}{\partial t} + \dfrac{\varepsilon\mu - 1}{c^2}\dfrac{\partial}{\partial t}\left(\dfrac{\partial \mathbf{u}}{\partial t}\times \mathbf{H}\right), \\[2ex]
\operatorname{rot} \mathbf{E} = -\dfrac{\mu}{c}\dfrac{\partial \mathbf{h}}{\partial t}, \\[2ex]
\mathrm{j} = \lambda_0\left[\mathbf{E} + \dfrac{\mu}{c}\left(\dfrac{\partial \mathbf{u}}{\partial t}\times \mathbf{H}\right)\right], \\[2ex]
\varrho\dfrac{\partial^2 \mathbf{u}}{\partial t^2} = G\nabla^2\mathbf{u} + (\lambda + G)\operatorname{grad}\operatorname{div}\mathbf{u} + \dfrac{\mu}{c}(\mathrm{j}\times \mathbf{H}) + \\[2ex]
\quad + \dfrac{1}{4\pi c}(\mu\varepsilon - 1)\left(\dfrac{\partial \mathbf{E}}{\partial t}\times \mathbf{H}\right) + \dfrac{\mu}{4\pi c^2}(\varepsilon\mu - 1)\dfrac{\partial}{\partial t}\left(\dfrac{\partial \mathbf{u}}{\partial t}\times \mathbf{H}\right)\times \mathbf{H} + \mathbf{P},
\end{cases}
$$

and

$$
\tag{2.2}
\operatorname{div} \mathbf{h} = 0, \quad \operatorname{div} \mathbf{D} = \varrho_e,
$$

where

$$
\tag{2.3}
\mathbf{D} = \varepsilon\left[\mathbf{E} + \frac{\mu\varepsilon - 1}{c\varepsilon}\frac{\partial \mathbf{u}}{\partial t}\times \mathbf{H}\right].
$$

On the basis of (2.1), (2.2) and (2.3) we have at the same time

$$
\tag{2.4}
\operatorname{div} \mathrm{j} = -\frac{\partial \varrho_e}{\partial t}.
$$

In the case of dielectrics these equations are simplified. We have, [3]:

$$
\tag{2.5}
\mathrm{j} = 0, \quad \varrho_e = 0.
$$

Then, the system of equations takes the form:

$$
\tag{2.6}
\begin{cases}
\operatorname{rot} \mathbf{h} = \dfrac{\varepsilon}{c}\dfrac{\partial \mathbf{E}}{\partial t} + \dfrac{\varepsilon\mu - 1}{c^2}\dfrac{\partial}{\partial t}\left(\dfrac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}\right), \\[2ex]
\operatorname{rot} \mathbf{E} = -\dfrac{\mu}{c}\dfrac{\partial \mathbf{h}}{\partial t}, \\[2ex]
\varrho \dfrac{\partial^2 \mathbf{u}}{\partial t^2} = G\nabla^2 \mathbf{u} + (\lambda + G) \operatorname{grad} \operatorname{div} \mathbf{u} + \dfrac{1}{4\pi c}(\mu\varepsilon - 1)\left(\dfrac{\partial \mathbf{E}}{\partial t}\right) \times \mathbf{H}) + \\[2ex]
\qquad\qquad + \dfrac{\mu}{4\pi c^2}(\varepsilon\mu - 1)\dfrac{\partial}{\partial t}\left(\dfrac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}\right) \times \mathbf{H} + \mathbf{P},
\end{cases}
$$

and

$$
\tag{2.7}
\operatorname{div} \mathbf{h} = 0, \qquad \operatorname{div} \mathbf{D} = 0.
$$

Hence

$$
\tag{2.8}
\operatorname{div} \mathbf{E} = -\frac{\mu\varepsilon - 1}{c\varepsilon} \operatorname{div}\left(\frac{\partial \mathbf{u}}{\partial t} \times \mathbf{H}\right).
$$

The introduction of the general form of the resolving functions in the system of equations (2.6) would, as was shown for an analogous system of equations in [1], enable the reduction of this system to separate partial differential equations of the 12th order for the resolving functions. However, proceeding as in [1], and introducing certain simplifications consisting in rejecting very weak couplings, the problem can be simplified considerably and reduced to the solution of equations of much lower orders, for the resolving functions.

With no limitation of the generality of the solution we can assume $\mathbf{H} = H_3 = H$. Then, eliminating $\mathbf{h}$ from (2.6), we obtain

$$
\tag{2.9}
\begin{cases}
\operatorname{rot}\operatorname{rot} \mathbf{E} + \dfrac{\mu\varepsilon}{c^2}\dfrac{\partial^2 \mathbf{E}}{\partial t^2} + \dfrac{\varepsilon\mu - 1}{c^3}\mu\left(\dfrac{\partial^3 \mathbf{u}}{\partial t^3} \times \mathbf{H}\right) = 0, \\[2ex]
K_1 u_1 - \dfrac{1}{\varrho}(\lambda + G)\dfrac{\partial}{\partial x_1} \operatorname{div} \mathbf{u} = \dfrac{1}{\varrho} P_1 + \nu \dfrac{\partial E_2}{\partial t}, \\[2ex]
K_1 u_2 - \dfrac{1}{\varrho}(\lambda + G)\dfrac{\partial}{\partial x_2} \operatorname{div} \mathbf{u} = \dfrac{1}{\varrho} P_2 - \nu \dfrac{\partial E_1}{\partial t}, \\[2ex]
K u_3 - \dfrac{1}{\varrho}(\lambda + G)\dfrac{\partial}{\partial x_3} \operatorname{div} \mathbf{u} = \dfrac{1}{\varrho} P_3,
\end{cases}
$$

where

$$
\tag{2.10}
\begin{cases}
K_1 = \dfrac{\partial^2}{\partial t^2}\left[1 + H^2 \dfrac{\mu(\varepsilon\mu - 1)}{4\pi c^2 \varrho}\right] - a_2^2 \nabla^2, \\[2ex]
K = \dfrac{\partial^2}{\partial t^2} - a_2^2 \nabla^2, \\[2ex]
\nu = \dfrac{\mu\varepsilon - 1}{4\pi c \varrho} H, \qquad a_2^2 = \dfrac{G}{\varrho}, \qquad \dfrac{c^2}{\mu\varepsilon} = c_0^2.
\end{cases}
$$

Making use of the relation

$$\text{rot rot } \mathbf{E} = \text{grad div} \mathbf{E} - \nabla^2 \mathbf{E}, \tag{2.11}$$

the system of equations (2.9) can be rewritten [bearing in mind (2.8)] in the form:

$$\left\{\begin{aligned} &NE_1 - \frac{\varepsilon\mu-1}{\varepsilon c}\frac{\partial^3 u_2}{\partial t^3}H - \frac{\mu\varepsilon-1}{\varepsilon\mu}\frac{\partial^2}{\partial x_1 \partial t}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\frac{Hc}{\varepsilon} = 0,\\ &NE_2 - \frac{\varepsilon\mu-1}{\varepsilon c}\frac{\partial^3 u_1}{\partial t^3}H - \frac{\mu\varepsilon-1}{\varepsilon\mu}\frac{\partial^2}{\partial x_2 \partial t}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\frac{Hc}{\varepsilon} = 0,\\ &NE_3 - \frac{\varepsilon\mu-1}{\varepsilon\mu}\frac{\partial^2}{\partial x_3 \partial t}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\frac{Hc}{\varepsilon} = 0;\end{aligned}\right. \tag{2.12}$$

$$\left\{\begin{aligned} &K_1 u_1 - \frac{\lambda+G}{\varrho}\frac{\partial}{\partial x_1}\operatorname{div}\mathbf{u} = \frac{1}{\varrho}P_1 + \nu\frac{\partial E_2}{\partial t},\\ &K_1 u_2 - \frac{\lambda+G}{\varrho}\frac{\partial}{\partial x_2}\operatorname{div}\mathbf{u} = \frac{1}{\varrho}P_2 - \nu\frac{\partial E_1}{\partial t},\\ &K u_3 - \frac{\lambda+G}{\varrho}\frac{\partial}{\partial x_3}\operatorname{div}\mathbf{u} = \frac{1}{\varrho}P_3,\end{aligned}\right. \tag{2.13}$$

where

$$N = \frac{\partial^2}{\partial t^2} - c_0^2\nabla^2. \tag{2.14}$$

In the operator K_1 (2.10) we have $H^2\mu(\varepsilon\mu-1)/(4\pi\varrho c^2) \ll 1$ and this term can be disregarded in relation to unity. Then

$$K_1 = K. \tag{2.15}$$

Eliminating E_1, E_2 from the last three of the equations (2.13), and substituting in the first three, we obtain:

$$\left\{\begin{aligned} &NKu_1 - \frac{\lambda+G}{\varrho}\frac{\partial}{\partial x_1}N\operatorname{div}\mathbf{u} - \frac{\varepsilon\mu-1}{\varepsilon c}\nu H\frac{\partial^4 u_1}{\partial t^4} -\\ &\qquad - \frac{\mu\varepsilon-1}{\varepsilon\mu}\nu\frac{\partial^3}{\partial x_2 \partial t^2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\frac{Hc}{\varepsilon} = \frac{1}{\varrho}NP_1 = P_1^*,\\ &NKu_2 - \frac{\lambda+G}{\varrho}\frac{\partial}{\partial x_2}N\operatorname{div}\mathbf{u} - \frac{\varepsilon\mu-1}{\varepsilon c}\nu H\frac{\partial^4 u_2}{\partial t^4} +\\ &\qquad + \frac{\mu\varepsilon-1}{\varepsilon\mu}\nu\frac{\partial^3}{\partial x_1 \partial t^2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\frac{Hc}{\varepsilon} = \frac{1}{\varrho}NP_2 = P_2^*,\\ &Ku_3 - \frac{\lambda+G}{\varrho}\frac{\partial}{\partial x_3}\operatorname{div}\mathbf{u} = \frac{1}{\varrho}P_3 = P_3^*;\end{aligned}\right. \tag{2.16}$$

$$NE_3 - \frac{\partial^2}{\partial x_3 \partial t}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\frac{\varepsilon\mu-1}{\varepsilon\mu}\frac{Hc}{\varepsilon} = 0. \tag{2.17}$$

Thus, similarly to [1], the original system of equations, for which the resolving functions were described by equations of the 12th order, has been reduced to the system of equations (2.16), which, after disjoining, reduces to an equation of the 10th order and the wave equation (2.17), from which E_3 is determined after determining **u** from the system of Eqs. (2.16). For the excitation force field, it is assumed that these forces admit the operation N. The system of equations (2.16) may in what follows be reduced to equations of order lower than the tenth. To this end, let us introduce, similarly to [1], the following set of resolving functions:

$$u_1 = \frac{\partial \Psi_1}{\partial x_1} - \frac{\partial \Psi_2}{\partial x_2}, \qquad u_2 = \frac{\partial \Psi_1}{\partial x_2} + \frac{\partial \Psi_2}{\partial x_1}, \qquad u_3 = \frac{\partial \Psi_3}{\partial x_3}. \tag{2.18}$$

Substituting (2.18) in the system of equations (2.16) we obtain:

$$\begin{cases} \dfrac{\partial}{\partial x_1}\left[\left(R - \dfrac{\lambda+G}{\varrho} N\nabla_1^2\right)\Psi_1 - \dfrac{\lambda+G}{\varrho} N \dfrac{\partial^2}{\partial x_3^2}\Psi_3\right] - \dfrac{\partial}{\partial x_2}\left(R + \gamma \dfrac{\partial^2}{\partial t^2}\nabla_1^2\right)\Psi_2 = P_1^*, \\ \dfrac{\partial}{\partial x_2}\left[\left(R - \dfrac{\lambda+G}{\varrho} N\nabla_1^2\right)\Psi_1 - \dfrac{\lambda+G}{\varrho} N \dfrac{\partial^2}{\partial x_3^2}\Psi_3\right] + \dfrac{\partial}{\partial x_1}\left(R + \gamma \dfrac{\partial^2}{\partial t^2}\nabla_1^2\right)\Psi_2 = P_2^*, \\ \dfrac{\partial}{\partial x_3}\left[-\dfrac{\lambda+G}{\varrho}\nabla_1^2\Psi_1 + \left(K - \dfrac{\lambda+G}{\varrho}\dfrac{\partial^2}{\partial x_3^2}\right)\Psi_3\right] = P_3^*, \end{cases} \tag{2.19}$$

where

$$\begin{cases} R = NK - \nu\eta \dfrac{\partial^4}{\partial t^4}, \qquad \nabla_1^2 = \dfrac{\partial^2}{\partial x_1^2} + \dfrac{\partial^2}{\partial x_2^2}, \\ \nu\eta = \dfrac{\varepsilon\mu - 1}{\varepsilon c} H\nu = \dfrac{(\varepsilon\mu-1)^2}{4\pi\varepsilon c^2 \varrho} H^2, \\ \gamma = \dfrac{\mu\varepsilon - 1}{\varepsilon^2\mu} Hc\nu = \dfrac{(\mu\varepsilon-1)^2}{4\pi\varrho\varepsilon^2\mu} H^2. \end{cases} \tag{2.20}$$

Representing P_i^* in the form

$$P_1^* = \frac{\partial X_1}{\partial x_1} - \frac{\partial X_2}{\partial x_2}, \qquad P_2^* = \frac{\partial X_1}{\partial x_2} + \frac{\partial X_2}{\partial x_1}, \qquad P_3^* = \frac{\partial X_3}{\partial x_3}, \tag{2.21}$$

where X_i is calculated, after transformations similar to [1], according to the equations

$$\begin{cases} X_1 = \displaystyle\iint_{\Omega_1} \left(P_1^* \frac{\partial}{\partial x_1} \ln\frac{1}{r} + P_2^* \frac{\partial}{\partial x_2} \ln\frac{1}{r}\right) d\xi_1 d\xi_2, \\ X_2 = \displaystyle\iint_{\Omega_1} \left(P_2^* \frac{\partial}{\partial x_1} \ln\frac{1}{r} - P_1^* \frac{\partial}{\partial x_2} \ln\frac{1}{r}\right) d\xi_1 d\xi_2, \\ X_3 = \displaystyle\int P_3^* d\xi_3, \\ r^2 = (x_1-\xi_1)^2 + (x_2-\xi_2)^2, \end{cases} \tag{2.22}$$

the system of equations (2.19) can be rewritten thus:

$$\left(R+\gamma\frac{\partial}{\partial t}\nabla_1^2\right)\Psi_2 = X_2, \tag{2.23}$$

$$\begin{cases} \left(R-\dfrac{\lambda+G}{\varrho}N\nabla_1^2\right)\Psi_1-\dfrac{\lambda+G}{\varrho}N\dfrac{\partial^2}{\partial x_3^2}\Psi_3 = X_1, \\ -\dfrac{\lambda+G}{\varrho}\nabla_1^2\Psi_1+\left(K-\dfrac{\lambda+G}{\varrho}\dfrac{\partial^2}{\partial x_3^2}\right)\Psi_3 = X_2. \end{cases} \tag{2.24}$$

The initial conditions for the system of equations (2.13) will be assumed in the form:

$$\begin{cases} u_k(x_1, x_2, x_3, 0) = f_k(x_1, x_2, x_3), \qquad \dfrac{\partial u_k(x_1, x_2, x_3, 0)}{\partial t} = g_k(x_1, x_2, x_3), \\ E_k(x_1, x_2, x_3, 0) = \dfrac{\partial E_k(x_1, x_2, x_3, 0)}{\partial t} = 0, \end{cases} \tag{2.25}$$

where f_k, g_k are assumed to be of the C^2 class.

The functions u_k in the system of equations (2.16) satisfy the following initial conditions:

$$\begin{cases} u_k\,(x_1, x_2, x_3, 0) = f_k(x_1, x_2, x_3,), \qquad \dfrac{\partial u_k(x_1, x_2, x_3, 0)}{\partial t} = g_k(x_1, x_2, x_3), \\ \dfrac{\partial^2 u_i(x_1, x_2, x_3, 0)}{\partial t^2} = \dfrac{\lambda+G}{\varrho}\dfrac{\partial}{\partial x_i}\operatorname{div}\mathbf{f}+a_2^2\nabla^2 f_i+\dfrac{1}{\varrho}P_i(x_1, x_2, x_3, 0) \qquad (i=1,2), \\ \dfrac{\partial^3 u_i(x_1, x_2, x_3, 0)}{\partial t^3} = \dfrac{\lambda+G}{\varrho}\dfrac{\partial}{\partial x_i}\operatorname{div}\mathbf{g}+a_2^2\nabla^2 g_i+\dfrac{1}{\varrho}\dfrac{\partial P_i(x_1, x_2, x_3, 0)}{\partial t} \\ \qquad\qquad (i = 1,2). \end{cases} \tag{2.26}$$

The functions ψ_i satisfy, according to the equations (2.21) the initial conditions

$$\begin{cases} \Psi_k(x_1, x_2, x_3, 0) = b_k(x_1, x_2, x_3), \\ \dfrac{\partial\Psi_k(x_1, x_2, x_3, 0)}{\partial t} = d_k(x_1, x_2, x_3) \qquad (k = 1,2,3), \\ \dfrac{\partial^2\Psi_j(x_1, x_2, x_3, 0)}{\partial t^2} = b_{1j}(x_1, x_2, x_3), \\ \dfrac{\partial^3\Psi_j(x_1, x_2, x_3, 0)}{\partial t^3} = d_{1j}(x_1, x_2, x_3) \qquad (j=1,2), \end{cases} \tag{2.27}$$

where b_k, d_k are determined from the equations

$$\begin{cases} b_1 = \displaystyle\iint_{\Omega_2}\left(f_1\frac{\partial}{\partial x_1}\ln\frac{1}{r}+f_2\frac{\partial}{\partial x_2}\ln\frac{1}{r}\right)d\xi_1 d\xi_2, \end{cases} \tag{2.28}$$

$$
(2.28)\qquad
\begin{cases}
b_2 = \iint\limits_{\Omega_3} \left(f_2 \frac{\partial}{\partial x_1} \ln \frac{1}{r} - f_1 \frac{\partial}{\partial x_2} \ln \frac{1}{r} \right) d\xi_1 d\xi_2, \\
b_3 = \int f_3 d\xi_3, \\
d_1 = \iint\limits_{\Omega_3} \left(g_1 \frac{\partial}{\partial x_1} \ln \frac{1}{r} + g_2 \frac{\partial}{\partial x_2} \ln \frac{1}{r} \right) d\xi_1 d\xi_2, \\
d_2 = \iint\limits_{\Omega_3} \left(g_2 \frac{\partial}{\partial x_1} \ln \frac{1}{r} - g_1 \frac{\partial}{\partial x_2} \ln \frac{1}{r} \right) d\xi_1 d\xi_2, \\
d_3 = \int g_3 d\xi_3.
\end{cases}
$$

The functions b_{1j}, d_{1j} may be calculated by means of the functions b_k, d_k on the basis of the relation (2.26).

Let us now disjoin the equations (2.24). We shall introduce new disjoining functions Φ_1, Φ_3:

$$
(2.29)\qquad
\begin{cases}
\Psi_1 = \left(K - \frac{\lambda+G}{\varrho} \frac{\partial^2}{\partial x_3^2} \right) \Phi_1 + \frac{\lambda+G}{\varrho} N \frac{\partial^2}{\partial x_3^2} \Phi_3, \\
\Psi_3 = \frac{\lambda+G}{\varrho} \nabla_1^2 \Phi_1 + \left(R - \frac{\lambda+G}{\varrho} N \nabla_1^2 \right) \Phi_3.
\end{cases}
$$

Substituting (2.29) in (2.24), we obtain:

$$
(2.30)\qquad \left[\left(R - \frac{\lambda+G}{\varrho} N \nabla_1^2 \right) \left(K - \frac{\lambda+G}{\varrho} \frac{\partial^2}{\partial x_3^2} \right) - \frac{(\lambda+G)^2}{\varrho^2} N \nabla_1^2 \frac{\partial^2}{\partial x_3^2} \right] \Phi_1 = X_1 \qquad (i = 1,3).
$$

These equations are apparently identical with those obtained in [1], it should however, be borne in mind that the symbols, which are identical in both equations, denote different expressions. The Eqs. (2.30) are simpler, in principle, than the corresponding equations of Ref. [1], because the difference between the operators K and K_1 vanishes.

The functions Φ_i satisfy the initial conditions:

$$
(2.31)\qquad
\begin{cases}
\Phi_k(x_1, x_2, x_3, 0) = \frac{\partial \Phi_k(x_1, x_2, x_3, 0)}{\partial t} = 0 \qquad (k = 1,3), \\
\frac{\partial^2 \Phi_1(x_1, x_2, x_3, 0)}{\partial t^2} = b_1(x_1, x_2, x_3,), \qquad \frac{\partial^3 \Phi_3(x_1, x_2, x_3, 0)}{\partial t^2} = 0, \\
\frac{\partial^3 \Phi_1(x_1, x_2, x_3, 0)}{\partial t^3} = d_1(x_1, x_2, x_3), \qquad \frac{\partial^2 \Phi_3(x_1, x_2, x_3, 0)}{\partial t^3} = 0, \\
\frac{\partial^4 \Phi_1(x_1, x_2, x_3, 0)}{\partial t^4} = b_{1,1}(x_1, x_2, x_3,) -
\end{cases}
$$

$$
(2.31)\quad \begin{cases} \qquad -\dfrac{\lambda+G}{\varrho}\dfrac{\partial^2}{\partial x_3^2}b_3(x_1,x_2,x_3)+\left(a_2^2\nabla^2+\dfrac{\lambda+G}{\varrho}\dfrac{\partial^2}{\partial x_3^2}\right)b_1(x_1,x_2,x_3),\\ \dfrac{\partial^4\Phi_3(x_1,x_2,x_3,0)}{\partial t^4}=b_3(x_1,x_2,x_3),\\ \dfrac{\partial^5\Phi_1(x_1,x_2,x_3,0)}{\partial t^5}=d_{1,1}(x_1,x_2,x_3)+\\ \qquad +\left(a_2^2\nabla^2+\dfrac{\lambda+G}{\varrho}\dfrac{\partial^2}{\partial x_3^2}\right)d_1(x_1,x_2,x_3)-\dfrac{\lambda+G}{\varrho}\dfrac{\partial^2}{\partial x_3^2}d_3(x_1,x_2,x_3),\\ \dfrac{\partial^5\Phi_3(x_1,x_2,x_3,0)}{dt^5}=d_3(x_1,x_2,x_3). \end{cases}
$$

The aim is thus reached. Similarly to the Ref. [1] for conductors, in the case of dielectrics the general equations of the 12th order for the resolving functions reduce to equations of the 2nd, 4th and 6th order ((2.17), (2.23) and (2.30), respectively). This essentially facilitates the solution of the problem, but the solution of the 6th order still remains difficult.

Similarly to [1], the equations reduce in the plane problem to the 4th order. Then $u_3 = E_3 = 0$, and we obtain, on the basis of (2.23) and (2.24):

$$
(2.32)\qquad \left(R+\gamma\frac{\partial^2}{\partial t^2}\nabla_1^2\right)\Psi_2 = X_2,\quad \left(R-\frac{\lambda+G}{\varrho}N\nabla_1^2\right)\Psi_1 = X_1.
$$

To avoid the solution of 6th order equations we shall consider the possibilities of simplification of the equations obtained for the resolving functions by rejecting certain small physical couplings of higher orders. This will enable the problem to be reduced to the solution of wave equations and one equation of the 4th order. The simplified equations thus obtained for coupled mechanical-electromagnetic fields in dielectrics are much simpler than analogous simplified equations of Ref. [1].

In the Ref. [1] two variants of simplification were considered. In the present paper only one variant will be considered—the other variant of the Ref. [1], concerning the dissipation.

3. Approximate Equations

The quantity $\nu\eta$ in the equations (2.23) and (2.30) can be rejected as a small quantity of a higher order. Similarly to [1], we shall show by means of an analogous one-dimensional example that $\nu\eta$ is of the order a_2^2/c^2, and may be disregarded in relation to unity independent of the time t at which the process of wave propagation is examined. Putting $\nu\eta = 0$ in the Eqs. (2.23) and (2.30), we shall obtain the following simplified forms of these equations:

$$
(3.1)\qquad \left(NK+\gamma\frac{\partial^2}{\partial t^2}\nabla_1^2\right)\Psi_2 = X_2,
$$

$$N\left[\left(K-\frac{\lambda+G}{\varrho}\nabla_1^2\right)\left(K-\frac{\lambda+G}{\varrho}\frac{\partial^2}{\partial x_3^2}\right)-\frac{(\lambda+G)^2}{\varrho^2}\frac{\partial^2}{\partial x_3^2}\nabla_1^2\right]\Phi_i = X_i \qquad (i = 1,3). \tag{3.2}$$

The equation (3.2) can also be rewritten thus

$$N\Box_1\Box_2\Phi_i = X_i \qquad (i = 1,3), \tag{3.3}$$

where

$$\begin{cases} \Box_1 = \dfrac{\partial^2}{\partial t^2}-a_1^2\nabla^2, \quad \Box_2 = \dfrac{\partial^2}{\partial t^2}-a_2^2\nabla^2 = K, \\ a_1^2 = \dfrac{\lambda+2G}{\varrho}, \qquad a_2^2 = \dfrac{G}{\varrho}. \end{cases} \tag{3.4}$$

Thus, the equations of the problem have been reduced to the solution of wave equations and one equation of the 4th order instead of the equations of the 2nd, 4th and 6th orders, in the case of $\nu\eta \neq 0$.

The Eqs. (3.3) are of the 6th order, but they are expressed by means of a triple product of wave operators which can be integrated in a simple manner by solving three wave equations succesively. Denoting

$$\Box_1\Box_2\Phi_i = \Theta_i, \tag{3.5}$$

we obtain from (3.3) the equation:

$$N\Theta_i = X_i, \tag{3.6}$$

where the initial conditions follow directly from (2.31).

After determining Θ_i from (3.6) and denoting

$$\Box_2\Phi_i = \Gamma_i, \tag{3.7}$$

Γ_i can be calculated from the equation

$$\Box_1\Gamma_i = \Theta_i \tag{3.8}$$

with appropriate initial conditions for Γ_i.

Γ_i being thus determined we calculate, from the wave equation (3.7) the functions Φ_i sought-for. Thus, finally, the problem is reduced to the known solution of the wave equation (2.17) and the wave equations (3.6), (3.7), (3.8) and an equation of the 4th order for ψ_2 (3.1).

The latter can be written in the form of a Volterra integro-differential equation:

$$\Psi_2 = \iint_{\Omega_t}\int G(\mathbf{x}-\xi, t-\tau)\left[X_2(\xi,\tau)-\gamma\frac{\partial^2}{\partial t^2}\nabla_1^2\Psi_2(\xi,\tau)\right]d\xi\, d\tau, \tag{3.9}$$

where G is Green's function for the equation

$$\Box_0\Box_2 G = \delta(\mathbf{x}-\xi, t), \tag{3.10}$$

where

$$\Box_0 = N.$$

The Eq. (3.10) will be solved, as before, by solving successively in a known manner two wave equations. The formulae which are known will not be cited here.

With the assumptions on X_i previously made the Eq. (3.9) has a convergent solution. If the original field H is small, γ may be treated as a small parameter and we can confine ourselves to the first approximation only. It will be shown now, on a simple, one dimensional case that $\nu\eta$ can be made equal to zero in the system of equations (2.13), which is equivalent to the rejection of a quantity of the order a_2^2/c^2 in relation to 1.

Similarly to [1] let us consider the case where $u_1 = u_3 = E_3 = E_2 = 0$, $P_1 = P_2 = P_3 = 0$. The initial conditions for E_1 will be assumed to be homogeneous and those for u_2 to have the form

$$u_2(x,0) = \varphi(x), \qquad \frac{\partial u_2(x,0)}{\partial t} = 0. \tag{3.11}$$

Then the system of equations (2.13) takes the following form, assuming that the solutions depend only on t and $x_1 = x$

$$\left\{\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} - c_0^2\frac{\partial^2}{\partial x^2}\right)E_1 = \gamma_1\frac{\partial^3 u_2}{\partial t\partial x_2^2} - \eta\frac{\partial^3 u_2}{\partial t^3},\\ &\left(\frac{\partial^2}{\partial t^2} - a_2^2\frac{\partial^2}{\partial x^2}\right)u_2 = -v\frac{\partial E_1}{\partial t}, \end{aligned}\right. \tag{3.12}$$

where

$$\gamma_1 = \frac{\mu\varepsilon - 1}{\mu\varepsilon^2}Hc, \qquad \eta = \frac{\varepsilon\mu - 1}{\varepsilon c}H, \qquad \nu = \frac{\mu\varepsilon - 1}{4\pi c\varrho}H. \tag{3.13}$$

where ν may also be considered to be a small parameter; therefore the above equation can be solved by means of the iteration method[(1)], by determining u_2 from the second equation, then E_1 from the first, u_2 from the second etc.

To make an appraisal of the right-hand sides of the first of the Eqs. (3.12), it suffices to substitute u_2 of the zero approximation and to set equal the two members. Hence their ratio is obtained.

The zero solution for $u_2(x,t)$ has the form:

$$u_2(x,t) = \frac{1}{2}[\varphi(x-a_2t)+\varphi(x+a_2t)]. \tag{3.14}$$

Substituting (3.13) in the right-hand member of the first of the equations (3.12) we have:

$$\gamma\frac{\partial^3 u_2}{\partial t\partial x^2} - \eta\frac{\partial^3 u_2}{\partial t^3} = \frac{1}{2}[(-\gamma a_2+\eta a_2^3)\varphi'''(x-a_2t)+(\gamma a_2-\eta a_2^3)\varphi'''(x+a_2t)]. \tag{3.15}$$

[(1)] The strong solution may also be found without difficulty but it is not necessary.

Setting equal the coefficients of $\varphi'''(x-a_2t)$ and $\varphi'''(x+a_2t)$ we have

$$\gamma a_2 \gg \eta a_2^3, \tag{3.16}$$

because, making use of (3.19),

$$\frac{\mu\varepsilon-1}{\mu\varepsilon^2} Hca_2 \gg \frac{\varepsilon\mu-1}{\varepsilon c} Ha_2^3 \tag{3.17}$$

Hence

$$\frac{c^2}{a_2^2\mu\varepsilon} = \frac{c_0^2}{a_2^2} \gg 1, \tag{3.18}$$

which confirms the assumption.

4. Resolving Functions for Anelastic Bodies

The considerations of the Sec. 2 are valid formally also in the case of anelastic bodies. The solutions being identical with those in the case of Ref. [1], we shall give final results for the resolving functions. As an example, a Boltzmann solid will be considered. It is assumed for simplicity that the initial conditions are homogeneous. Subjecting the system of equations (2.13), with additional relaxation terms for a Boltzmann body to the Laplace transformation and introducing, as in Sec. 2 the transforms of the resolving functions, which will be called, for brevity, operational resolving functions, we obtain:

$$(\bar{R}+\gamma p^2\nabla_1^2) = \bar{X}_2, \tag{4.1}$$

$$\left[\left(\bar{R}-\frac{\bar{\lambda}+\bar{G}}{\varrho}\bar{N}\nabla_1^2\right)\left(\bar{K}-\frac{\bar{\lambda}+\bar{G}}{\varrho}\frac{\partial^2}{\partial x_3^2}\right)-\frac{(\bar{\lambda}+\bar{G})^2}{\varrho^2}\bar{N}\nabla_1^2\frac{\partial^2}{\partial x_3^2}\right]\bar{\Phi}_i = \bar{X}_i \quad (i=1,3), \tag{4.2}$$

or, assuming $\eta = 0$,

$$(\bar{N}\bar{K}+\gamma p^2\nabla_1^2)\bar{\psi}_2 = X_2, \tag{4.3}$$

$$\bar{N}\overline{\Box}_1\overline{\Box}_2\bar{\Phi}_i = \bar{X}_i \qquad (i=1,3), \tag{4.4}$$

where the barred operators denote the operators defined in the foregoing section except that $\partial/\partial t$ is replaced by p, λ, G— by $\bar{\lambda}$, $\bar{G}$, and

$$\bar{\lambda} = \lambda-\bar{R}(p), \qquad \bar{G} = G-\bar{Q}(p), \tag{4.5}$$

where $\bar{R}(p)$, $\bar{Q}(p)$ are the transforms of the relaxation functions appearing in the equation

$$\sigma_{ik} = \lambda \operatorname{div}\mathbf{u}\,\delta_{ik}+2G\varepsilon_{ik}-\int_0^t [R(t-\tau)\operatorname{div}\mathbf{u}(\tau)\delta_{ik}+2Q(t-\tau)\varepsilon_{ik}(\tau)]d\tau. \tag{4.6}$$

The solvability of these equations is, depending on the relaxation functions, similar to the theory of anelastic bodies. This problem will not be dealt with here.

5. Conclusion

The present paper and Ref. [1], enable the construction of a solution for combined mechanical-electromagnetic fields in a relatively simple way for real conductors and dielectrics.

In these cases, the general equations of the 12th order for the resolving functions can be reduced to equations of the 2nd and 4th order, for which the solutions of the Cauchy problem are in most cases known in the theory of elasticity. This considerably simplifies the solution. In Ref. [2], similar solutions (the equations of the 2nd and 4th order being solved in quadratures) are obtained for a perfect conductor.

If the conduction and displacement currents are of the same order the above solution cannot be directly used, and that problem requires separate treatment.

References

[1] S. Kaliski, *Cauchy's problem for a real isotropic elastic conductor in a magnetic field*, Proc. Vibr. Probl., 2, 2 (1961).

[2] S. Kaliski, *Cauchy's problem for a perfect isotropic and transversally isotropic conductor in magnetic field*, Proc. Vibr. Probl. 2, 2 (1961).

[3] И. Е. Тамм, *Основы теории электричества*, Москва 1957.

Streszczenie

PROBLEM CAUCHY'EGO DLA SPRĘŻYSTYCH DIELEKTRYKÓW W POLU MAGNETYCZNYM

W pracy rozważony został problem Cauchy'ego dla odkształcalnego dielektryka sprężystego i niesprężystego znajdującego się w pierwotnym, stałym polu magnetycznym. Podobnie jak w [1] udało się w przypadku wymuszeń polami mechanicznymi rozbić ogólne równania cząstkowe 12 rzędu dla funkcji rozwiązujących na szereg równań falowych i jedno równanie 4 rzędu, które z kolei sprowadzono do równania różniczkowo-całkowego typu Volterry z jądrem będącym kombinacją rozwiązań równań falowych. Tak więc zagadnienie zostało sprowadzone w zasadzie, do kombinacji rozwiązań znanych. Rozwiązania otrzymane dla dielektryków opierają się ze względu na podobny charakter równań wyjściowych na analogicznej konstrukcji funkcji rozwiązujących jak w [1] z tym, że rozwiązania dla dielektryków kształtują się znacznie prościej aniżeli dla przewodników ze względu na inny charakter uproszczeń równań wyjściowych.

Резюме

ЗАДАЧА КОШИ ДЛЯ УПРУГИХ ДИЭЛЕКТРИКОВ В МАГНИТНОМ ПОЛЕ

Рассматривается задача Коши для деформируемого упругого и неупругого диэлектрика, находящегося в первичном постоянном магнитном поле. По-

добно, как в работе [1], удалось, в случае вынуждений механическим полем, разбить общее уравнение в частных производных 12-го порядка для разрешающих функций на ряд волновых уравнений и одно уравнение 4-го порядка, которое в свою очередь сведено к интегро-дифференциальному уравнению типа Вольтери, с ядром являющимся комбинацией решений волновых уравнений. Таким образом задача сводится в основном к комбинации известных решений. Решения полученные для диэлектриков, ввиду подобного решения исходных уравнений, основываются на аналогичной конструкции разрешающих функций как в [1] с тем, что решения для диэлектриков по сравнению с проводниками, имеют более простую форму, ввиду иного характера упрощений исходных уравнений.

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THE VIBRATION OF A CYLINDRICAL SHELL OF FINITE LENGTH WITH A SUPERSONIC INSIDE FLOW

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Introduction

A number of rocket engineering problems—those of rocket and ramjet nozzles for instance — consist in determining the conditions of self-excited vibrations (flutter). The physical model for these problems is a cylindrical or axially symmetric shell of finite length acted on by a gas flowing inside the shell. The way of fixing the shell (the boundary conditions) and the so-called structural damping is determined by the design features and the so-called material damping — by the kind of material used. Each one of these factors reduces, in general, the possibility of vibrations, but the appearance of them is not entirely prevented, however.

In the literature there are some works concerning the flutter problem of infinitely long shells flown past a supersonic flow outside the shell, [1], [2], [3], [4], [5], [6]. Material and structural damping and also the anisotropy of the material are taken into account in [5] and [6] alone. For the analysis of the flutter problem it is important to have the high degree of accuracy of determining the gas pressure on the shell, provided that the possibility of actual computation is not impaired by mathematical complication.

In the recent works [7] and [8], considerable simplifications are made for the aerodynamic forces, or only a special vibration is considered (axially symmetric vibrations).

In the present paper a flow inside a shell is considered on the basis of the linearized theory of flow and the equations of the shell are deduced according to the general bending theory of shells (cf. [9], [10]). The flutter problem of a shell of finite length requires the satisfaction of the boundary conditions for the potential of flow and the equations of motion of the shell. This complicates the problem from the qualitative and numerical point of view.

The solution presented enables actual numerical computation. The computation algorithm proposed may be adapted for computation by means of electronic computers.

The method of analysis of the influence of each particular structural and aerodynamic parameters (diameter, thickness and length of the shell), the material damping coefficient, the flow velocity of the gas etc. — is based on the Neumark-Nyquist method which enables convenient graphical representation of the results [13], [23].

1. The Statement of the Problem

Let us consider a cylindrical shell of finite length the radius of the middle surface being R and the wall-thickness h (cf. Fig. 1).

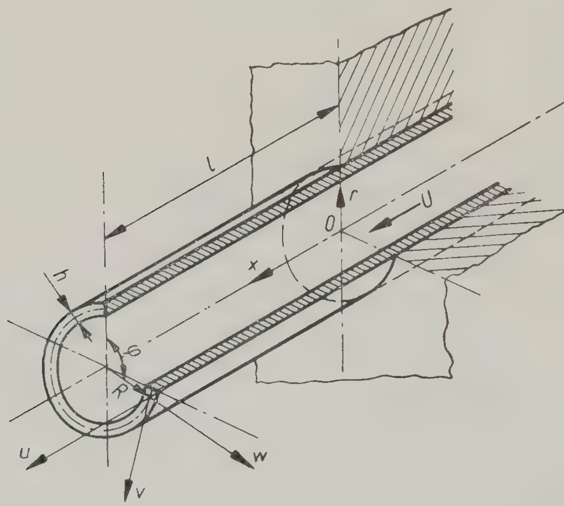

Fig. 1

The flutter phenomenon of the shell will be investigated for a gas flow through the shell of which the velocity of the unperturbed supersonic flow is U. The material damping of the shell will be taken into consideration by assuming the Voigt model, similarly to the Ref. [5]. The stress-strain relation can be assumed in the form

$$\sigma_x = \left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\Delta+2\left(\mu_1+\mu_2\frac{\partial}{\partial t}\right)\varepsilon_x,$$

$$\sigma_\varphi = \left(\lambda_1+\lambda_2\frac{\partial}{\partial t}\right)\Delta+2\left(\mu_1+\mu_2\frac{\partial}{\partial t}\right)\varepsilon_\varphi,$$

$$\tau_{x\varphi} = \left(\mu_1+\mu_2\frac{\partial}{\partial t}\right)\varepsilon_{x\varphi},$$

where $\Delta = \varepsilon_x + \varepsilon_\varphi$, ε_x, ε_φ—unit elongations in the directions of the axes u, v (Fig. 1), $\varepsilon_{x\varphi}$—change of the angle between the axes u and v,

$$\lambda_1 = \frac{E\nu}{1-\nu^2}, \qquad \mu_1 = \frac{E}{2(1+\nu)},$$

where: λ_2—volume damping coefficient, μ_2—distortion damping coefficient, E—Young's modulus, ν—Poisson's ratio.

On the basis of the general bending theory of shells, [9], we obtain the equilibrium equations in the form

$$(1.1)\quad \begin{cases} (\lambda+2\mu)\dfrac{\partial^2 u}{\partial x^2} + (1+c_1^2)\mu\dfrac{\partial^2 u}{\partial \varphi^2} + (\lambda+\mu)\dfrac{\partial^2 v}{\partial x \partial \varphi} + \lambda\dfrac{\partial w}{\partial x} - \\ \qquad - c^2(\lambda+2\mu)\dfrac{\partial^3 w}{\partial x^3} + c^2\mu\dfrac{\partial^3 w}{\partial x \partial \varphi^2} - R\varrho_s\dfrac{\partial^2 u}{\partial t^2} = 0, \\ \left(\lambda+\mu+\dfrac{9}{5}c^2\mu\right)\dfrac{\partial^2 u}{\partial x \partial \varphi} + (1+3c_1^2)\mu\dfrac{\partial^2 v}{\partial x^2} + (\lambda+2\mu)\dfrac{\partial^2 v}{\partial \varphi^2} + (\lambda+2\mu)\dfrac{\partial w}{\partial \varphi} - \\ \qquad - \left[\left(3c^2 - \dfrac{9}{5}c^4\right)\mu + c^2\lambda\right]\dfrac{\partial^3 w}{\partial x^2 \partial \varphi} - R\varrho_s\dfrac{\partial^2 v}{\partial t^2} = 0, \\ \left(c^2 - \dfrac{9}{5}c^4\right)\mu\dfrac{\partial^3 u}{\partial x \partial \varphi^2} - (\lambda+2\mu)c^2\dfrac{\partial^3 u}{\partial x^3} - c^2(\lambda+3\mu)\dfrac{\partial^3 v}{\partial x^2 \partial \varphi} + \\ \qquad + c^2(\lambda+2\mu)\left(\dfrac{\partial^4 w}{\partial x^4} + 2\dfrac{\partial^4 w}{\partial x^2 \partial \varphi^2} + \dfrac{\partial^4 w}{\partial \varphi^4}\right) - \dfrac{9}{5}c^4(\lambda+\mu)\dfrac{\partial^4 w}{\partial x^2 \partial \varphi^2} - \\ \qquad - \dfrac{9}{5}c^4\lambda\dfrac{\partial^2 w}{\partial x^2} + 2c^2(\lambda+2\mu)\dfrac{\partial^2 w}{\partial \varphi^2} + \lambda\dfrac{\partial u}{\partial x} + (\lambda+2\mu)\dfrac{\partial v}{\partial \varphi} + \\ \qquad + (1+c^2)(\lambda+2\mu)w + R^2\varrho_s\dfrac{\partial^2 w}{\partial t^2} - \dfrac{R^2}{h}p = 0 \ (^1) \end{cases}$$

where

$c^2 = \dfrac{h^2}{12R^2}, \quad c_1 = c,$

u, v, w displacements of the middle surface in the direction of the respective axes (Fig. 1),

p difference of gas pressure of perturbated flow and unperturbated one inside the shell,

ϱ_s mass density of the shell,

ϱ mass density of the gas.

(1) We assume that the pressure difference between the unperturbated flow internal and the external (immobile) gas can be disregarded as well as the influence of the radiation on damping.

In his theory of cylindrical shells Vlasov, [9], rejects the terms containing c^4 and c_1^2. Flügge, [10], rejects only those containing c^4, the terms containing c_1^2 are retained. In the present considerations the terms containing c^4 will be the only ones rejected(²).

We shall consider a shell rigidly fixed at one end, the other end being free (Fig. 1) The fixed end conditions are therefore such

$$u = v = w = \frac{\partial w}{\partial x} = 0 \qquad \text{for} \quad x = 0. \tag{1.2}$$

For the free end we have:

$$\begin{cases} T_{11} = 0, \quad T_{12} + \dfrac{M_{12}}{R} = 0 & \text{for} \quad x = l, \\ M_{11} = 0, \quad N_1 - \dfrac{1}{R}\dfrac{\partial M_{12}}{\partial \varphi} = 0 & \text{for} \quad x = l. \end{cases} \tag{1.3}$$

The symbols for moments and forces are given in Fig. 2.

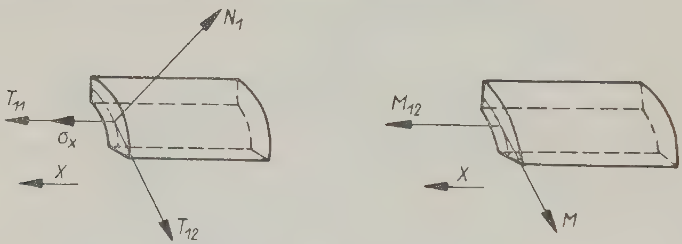

Fig. 2

The conditions (1.3) may be expressed in the displacements u,v,w:

$$\begin{cases} (\lambda+2\mu)\dfrac{\partial u}{\partial x} + \lambda\dfrac{\partial v}{\partial \varphi} + \lambda w - c^2(\lambda+2\mu)\dfrac{\partial^2 w}{\partial x^2} = 0, \\ \mu\dfrac{\partial u}{\partial \varphi} + (1+3c^2)\mu\dfrac{\partial v}{\partial x} - 3c^2\mu\dfrac{\partial^2 w}{\partial x \partial \varphi} = 0, \\ (\lambda+2\mu)\dfrac{\partial^2 w}{\partial x^2} + \lambda\dfrac{\partial^2 w}{\partial \varphi^2} - (\lambda+2\mu)\dfrac{\partial u}{\partial x} - \lambda\dfrac{\partial v}{\partial \varphi} = 0, \\ (\lambda+2\mu)\dfrac{\partial^2 u}{\partial x^2} - \mu\dfrac{\partial u}{\partial \varphi^2} - (\lambda+\mu)\dfrac{\partial^2 w}{\partial x \partial \varphi} - (\lambda+2\mu)\dfrac{\partial^3 w}{\partial x^3} - \lambda\dfrac{\partial^3 w}{\partial x \partial \varphi^2} = 0. \end{cases} \tag{1.4}$$

The inside gas flow under consideration is acted on by the vibrating shell. The equation of the potential Φ of a linearized flow has, in cylindrical coordinates, the form:

$$a^2\Delta\Phi = \frac{\partial^2\Phi}{\partial t^2} + 2U\frac{\partial^2\Phi}{\partial x \partial t} + U^2\frac{\partial^2\Phi}{\partial x^2}, \tag{1.5}$$

(²) The influence of the c and c_1 is appraised numerically to some extent in Ref. [14].

where

$$\frac{\partial \Phi}{\partial x} = \varrho (V_x - U), \qquad \frac{1}{r}\frac{\partial \Phi}{\partial \varphi} = \varrho V_\varphi, \qquad \frac{\partial \Phi}{\partial r} = \varrho V_r,$$

a — velocity of sound in gas, U — undisturbed flow velocity, V_x, V_φ, V_r — velocity components in the respective directions.

The potential $\Phi(x, r, \varphi, t)$ will be sought for in the form

$$\Phi(x, r, \varphi, t) = K(r)\Phi_1(x, \varphi, t).$$

From the boundary condition of flow on the wall of the shell

(1.6) $$V_r = \frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} V_x + \frac{1}{R}\frac{\partial w}{\partial \varphi} V_\varphi \, (^3)$$

relating the potential with the displacements and assuming

$$K'(R) = \left[\frac{dK(r)}{dr}\right]_{r=R} \neq 0$$

after some simple manipulations, we obtain

(1.7) $$\Phi(x, r, \varphi, t) = \frac{K(r)}{K'(R)} \varrho(x, R, \varphi, t)\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right).$$

Hence, assuming ϱ to be constant, we obtain

(1.8) $$p = -\frac{K(R)}{K'(R)}\varrho\left(\frac{\partial^2 w}{\partial t^2} + 2U\frac{\partial^2 w}{\partial x \partial t} + U^2\frac{\partial^2 w}{\partial x^2}\right).$$

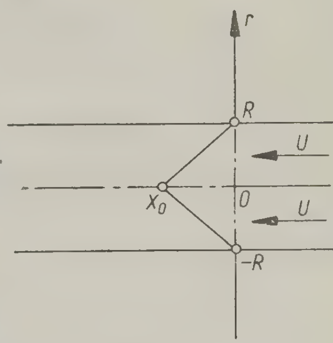

Fig. 3

If in the boundary condition for the potential the fact is accounted for that the perturbation potential of supersonic flow, $M_a = U/c > 1$, we have on the conical surface $[x = x_0 - (x_0/R)r]$ with the apex at the point x_0, that the perturbation potential has the form (Fig. 3)

(1.9) $$\Phi = 0.$$

2. The Solution Method

Particular solutions of the system of equations (1.1) will be sought for in the form:

(2.1) $$\begin{cases} u = Ae^{i(\alpha x + n\varphi + \omega t)}, \\ v = Be^{i(\alpha x + n\varphi + \omega t)}, \\ w = Ce^{i(\alpha x + n\varphi + \omega t)}. \end{cases}$$

On the basis of the Eq. (1.7), we shall search the potential Φ of the form:

(2.2) $$\Phi(x, r, \varphi, t) = K(r)e^{i(\alpha x + n\varphi + \omega t)}.$$

(³) In further considerations we assume $V_r = \dfrac{\partial w}{\partial t} + \dfrac{\partial w}{\partial x} U$.

Substituting (2.2) in the equation of the potential (1.5) we find that the function $K(r)$ satisfies the Bessel equation

$$K''(r)+\frac{1}{r}K'(r)-\left(m^2+\frac{n^2}{r^2}\right)K(r)=0, \tag{2.3}$$

where

$$m^2=\alpha^2\left(1-\frac{U^2}{a^2}\right)-\frac{\omega^2+2U\alpha\omega}{a^2}. \tag{2.4}$$

The general integral of the Eq. (2.3) is:

$$K(r)=\bar{c}_1N_n(imr)+c_2I_n(imr).$$

Hence, in view of the boundedness of the potential Φ for $r=0$ we obtain $\bar{c}_1=0$, or

$$K(r)=c_2I_n(imr), \tag{2.5}$$

where I_n is a Bessel function of the nth order of an complex variable.

On the basis of (2.5), we can determine the ratio $K(R)/K'(R)$ appearing in the relation (1.8).

Substituting (1.8) and (2.1) in the system of equations (1.1), we obtain a system of homogeneous linear equations for A, B, C,:

$$a_{j,1}A+a_{j,2}B+a_{j,3}C=0 \qquad (j=1,2,3), \tag{2.6}$$

where $a_{j,1}$, $a_{j,2}$, $a_{j,3}$ are

$$\left\{\begin{aligned}
a_{1,1}&=\alpha^2(\theta_1+\theta_2\omega i)+n^2(1+c^2)(\mu_1+\mu_2\omega i)-R^2\varrho_s\omega^2,\\
a_{1,2}&=\alpha n[(\lambda_1+\mu_1)+(\lambda_2+\mu_2)\omega i],\\
a_{1,3}&=-i\alpha[\lambda_1+\lambda_2\omega i+c^2\alpha^2(\theta_1+\theta_2\omega i)-c^2n^2(\mu_1+\mu_2\omega i)],\\
a_{2,1}&=\alpha n[(\lambda_1+\mu_1)+(\lambda_2+\mu_2)\omega i],\\
a_{2,2}&=(1+3c^2)(\mu_1+\mu_2\omega i)\alpha^2+n^2(\theta_1+\theta_2\omega i)-R^2\varrho_s\omega^2,\\
a_{2,3}&=-in\{(\theta_1+\theta_2\omega i)+\alpha^2c^2[(\lambda_1+3\mu_1)+(\lambda_2+3\mu_2)\omega i]\},\\
a_{3,1}&=i\alpha n^2c^2(\mu_1+\mu_2\omega i)-c^2\alpha^3i(\theta_1+\theta_2\omega i)-\alpha i(\lambda_1+\lambda_2\omega i),\\
a_{3,2}&=-in\{c^2\alpha^2[(\lambda_1+3\mu_1)+(\lambda_2+3\mu_2)\omega i]+\theta_1+\theta_2\omega i\},\\
a_{3,3}&=-c^2(\theta_1+\theta_2\omega i)(\alpha^2+n^2)-2c^2n^2(\theta_1+\theta_2\omega i)+\\
&\quad+(1+c^2)(\theta_1+\theta_2\omega i)-R^2\varrho_s\omega^2-R^2\frac{K(R)}{hK'(R)}\varrho(\omega^2+2U\alpha\omega+U^2\alpha^2)=0,
\end{aligned}\right. \tag{2.7}$$

where

$$\theta_1=\lambda_1+2\mu_1, \qquad \theta_2=\lambda_2+2\mu_2.$$

In order that the system of equations (2.6) should have a non-trivial solution the characteristic determinant W of this system must be zero or, in other words:

$$W=\operatorname{Det}(a_{i,j})=0. \tag{2.8}$$

It is easy to observe from (2.7) that $a_{i,j} = a_{j,i}$ $(i,j = 1, 2, 3)$. The determinant (2.8) depends on α and ω; therefore the Eq. (2.8) has the form:

$$W(\alpha,\omega) = 0 \ (^4). \tag{2.9}$$

The Eq. (2.9) contains two quantities of interest. These are: α — the deformation wave length and ω — the angular frequency of vibration. To determine them the boundary conditions of the shell must be used (cf. § 3 B).

It is impossible, in general, to find directly the roots α_j of the Eq. (2.9) in terms of ω, U. The establishment of two equations each containing α_j or ω only will be the object of a further part of the paper (Sec. 3B).

3. The Boundary Conditions

A. The Boundary Conditions of the Potential. The potential of flow Φ [cf. (1.8)] must satisfy the Eq. (1.5) and the boundary condition (1.9).

It can easily be shown that if $\Phi = \Phi(\alpha_j)$ satisfies the Eq. (1.5) and (1.6) for $r = R$, the expression

$$\Phi = \sum_{j=1}^{\infty} D_j \Phi_j \tag{3.1}$$

will also be a solution. The constants D_j are choosen to satisfy the conditions (1.9).

Substituting for this purpose (3.1) in the boundary condition (1.9) we obtain, on the basis of (2.2) and (2.5), the expression

$$[\Phi]_{x=x_0\left(1-\frac{r}{R}\right)} = \sum_{j=1}^{\infty} D_j I_n(im_j r) e^{i\alpha_j (x_0 - x_0 r/R)} = 0. \tag{3.2}$$

Expanding the functions $I_n(im_j r)$ and $e^{i\alpha_j x_0(1-r/R)}$ in power series of r we collect like powers(5) of r. Setting the coefficients equal to zero we obtain an infinite linear system of algebraic equations in D_j

$$\left\{\begin{array}{l} \sum_{j=1}^{\infty} D_j a_1(\alpha_j) b_1(\alpha_j) = 0, \\ \sum_{j=1}^{\infty} D_j [a_1(\alpha_j) b_2(\alpha_j) + a_2(\alpha_j) b_1(\alpha_j)] = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=1}^{\infty} D_j [a_1(\alpha_j) b_m(\alpha_j) + a_2(\alpha_j) b_{m-1}(\alpha_j) + \ldots + a_m(\alpha_j) b_1(\alpha_j)] = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right. \tag{3.3}$$

(4) It can easily be shown on the basis of what is known as Picard's minor theorem (cf[11]) that the equation has an infinite number of roots α_j $(j = 1,2,\ldots)$.

(5) The process of grouping these terms is admissible by virtue of the Mertens theorem (cf. (cf. [12]).

where $a_\nu(\alpha_j)$ and $b_\xi(\alpha_j)$ are coefficients of terms of expansion in power series of the functions $I_n(im_jr)$ and $e^{i\alpha_j x_0(1-r/R)}$.

The expressions

$$(3.4)\qquad \begin{cases} u = \sum\limits_{\nu=1}^{\infty} A_\nu e^{i(\alpha_\nu x+n\varphi+\omega t)}, \\ v = \sum\limits_{\nu=1}^{\infty} B_\nu e^{i(\alpha_\nu x+n\varphi+\omega t)}, \\ w = \sum\limits_{\nu=1}^{\infty} C_\nu e^{i(\alpha_\nu x+n\varphi+\omega t)}, \end{cases}$$

and the relation (3.1) satisfy the system of equations (1.1)([6]). The constants A_ν, B_ν, C_ν (which have not yet been determined) satisfy the system of equations (2.6) The constants A_ν, B_ν, C_ν will be determined by means of the boundary conditions of the shell.

B. The Boundary Conditions of the Shell. On the basis of the first two of the Eqs. (2.6), A_ν and B_ν can be expressed in terms of C_ν:

$$A_\nu = f(\alpha_\nu)C_\nu, \qquad B_\nu = g(\alpha_\nu)C_\nu.$$

Hence, by virtue of (3.4), we obtain:

$$(3.5)\qquad \begin{cases} u = \sum\limits_{\nu=1}^{\infty} f(\alpha_\nu)C_\nu e^{i(\alpha_\nu x+n\varphi+\omega t)}, \\ v = \sum\limits_{\nu=1}^{\infty} g(\alpha_\nu)C_\nu e^{i(\alpha_\nu x+n\varphi+\omega t)}, \\ w = \sum\limits_{\nu=1}^{\infty} C_\nu e^{i(\alpha_\nu x+n\varphi+\omega t)}. \end{cases}$$

Substituting (3.1) in the boundary conditions (1.2) and (1.4), we obtain eight equations

$$(3.6)\qquad \begin{cases} \sum\limits_{\nu=1}^{\infty} f(\alpha_\nu)C_\nu = 0, & \sum\limits_{\nu=1}^{\infty} C_\nu = 0, \\ \sum\limits_{\nu=1}^{\infty} g(\alpha_\nu)C_\nu = 0, & \sum\limits_{\nu=1}^{\infty} \alpha_\nu C_\nu = 0; \end{cases}$$

$$(3.7)\qquad \begin{cases} \sum\limits_{\nu=1}^{\infty} [(\theta_1+\theta_2\omega i)\alpha_\nu f(\alpha_\nu)+(\lambda_1+\lambda_2\omega i)nig(\alpha_\nu)+ \\ \qquad\qquad +(\lambda_1+\lambda_2\omega i)+c^2(\theta_1+\theta_2\omega i)\alpha_\nu^2]C_\nu e^{i\alpha_\nu l} = 0, \\ \sum\limits_{\nu=1}^{\infty} [(\theta_1+\theta_2\omega i)\alpha_\nu^2+(\lambda_1+\lambda_2\omega i)n^2+(\theta_1+\theta_2\omega i)\alpha_\nu if(\alpha_\nu)+ \\ \qquad\qquad +(\lambda_1+\lambda_2\omega i)nig(\alpha_\nu)]C_\nu e^{i\alpha_\nu l} = 0, \end{cases}$$

([6]) See also Sec. 5.

$$\left\{\begin{aligned}&\sum_{\nu=1}^{\infty}[f(\alpha_\nu)ni+g(\alpha_\nu)(1+3c^2)\alpha_\nu i+3c^2\alpha_\nu n]C_\nu e^{i\alpha_\nu l}=0,\\&\sum_{\nu=1}^{\infty}[(\theta_1+\theta_2\omega i)\alpha_\nu^2 f(\alpha_\nu)-(\mu_1+\mu_2\omega i)n^2 f(\alpha_\nu)+(\lambda_1+\lambda_2\omega i-\mu_1-\mu_2\omega i)n\alpha_\nu g(\alpha_\nu)-\\&\qquad -(\theta_1+\theta_2\omega i)\alpha_\nu^3 i-(\lambda_1+\lambda_2\omega i)\alpha_\nu n^2 i]C_\nu e^{i\alpha_\nu l}=0.\end{aligned}\right.$$

Making use of (1.6) the constants D_j may be expressed in function of C_j:

$$D_j=C_j\frac{(\omega+a_jU)\varrho}{m_jI_n'(im_jR)}. \tag{3.8}$$

We introduce the notation $\tilde{C}_j=j!C_j$.

From (3.6), (3.7), (3.3), (3.8) we obtain an infinite system of equations in the constants $\tilde{C}_j$:

$$\left\{\begin{aligned}&\sum_{\nu=1}^{\infty}\frac{1}{\nu!}f(\alpha_\nu)\tilde{C}_\nu=0,\qquad \sum_{\nu=1}^{\infty}\frac{1}{\nu!}\tilde{C}_\nu=0,\\&\sum_{\nu=1}^{\infty}\frac{1}{\nu!}g(\alpha_\nu)\tilde{C}_\nu=0,\qquad \sum_{\nu=1}^{\infty}\frac{\alpha_\nu}{\nu!}\tilde{C}_\nu=0,\\&\sum_{\nu=1}^{\infty}[(\theta_1+\theta_2\omega i)\alpha_\nu f(\alpha_\nu)+(\lambda_1+\lambda_2\omega i)nig(\alpha_\nu)+(\lambda_1+\lambda_2\omega i)+\\&\qquad +c^2(\theta_1+\theta_2\omega i)\alpha_\nu^2]\frac{1}{\nu!}\tilde{C}_\nu e^{i\alpha_\nu l}=0,\\&\sum_{\nu-1}^{\infty}[(\theta_1+\theta_2\omega i)\alpha_\nu^2+(\lambda_1+\lambda_2\omega i)n^2+(\theta_1+\theta_2\omega i)\alpha_\nu if(\alpha_\nu)+\\&\qquad +(\lambda_1+\lambda_2\omega i)\,\mathrm{nig}\,(\alpha_\nu)]\frac{1}{\nu!}\tilde{C}_\nu e^{i\alpha_\nu l}=0,\\&\sum_{\nu=1}^{\infty}[f(\alpha_\nu)ni+g(\alpha_\nu)(1+3c^2)\alpha_\nu i+3c^2\alpha_\nu n]\frac{1}{\nu!}\tilde{C}_\nu e^{i\alpha_\nu l}=0,\\&\sum_{\nu=1}^{\infty}[(\theta_1+\theta_2\omega i)\alpha_\nu^2 f(\alpha_\nu)-(\mu_1+\mu_2\omega i)n^2 f(\alpha_\nu)+(\lambda_1+\lambda_2\omega i-\mu_1-\mu_2\omega i)n\alpha_\nu g(\alpha_\nu)-\\&\qquad -(\theta_1+\theta_2\omega i)\alpha_\nu^3 i-(\lambda_1+\lambda_2\omega i)\alpha_\nu n^2 i]\frac{1}{\nu!}\tilde{C}_\nu e^{i\alpha_\nu l}=0,\\&\sum_{\nu=1}^{\infty}\frac{(\omega+U\alpha_\nu)\varrho}{m_\nu I_n'(m_\nu iR)\nu!}a_1(\alpha_\nu)b_1(\alpha_\nu)\tilde{C}_\nu=0,\\&\sum_{\nu=1}^{\infty}\frac{(\omega+U\alpha_\nu)\varrho}{m_\nu I_n'(m_\nu iR)\nu!}[a_1(\alpha_\nu)b_2(\alpha_\nu)+a_2(\alpha_\nu)b_1(\alpha_\nu)]\tilde{C}_\nu=0,\\&\dots\dots\dots\dots\dots\dots\dots\dots\\&\sum_{\nu=1}^{\nu\infty}\frac{(\omega+U\alpha_\nu)\varrho}{m_\nu I_n'(m_\nu iR)\nu!}[a_1(\alpha_\nu)b_k(\alpha_\nu)+a_2(\alpha_\nu)b_{k-1}(\alpha_\nu)+\dots+a_k(\alpha_\nu)b_1(\alpha_\nu)]\tilde{C}_\nu=0,\\&\dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}\right. \tag{3.9}$$

Let us denote the coefficients of $\tilde{C}_\nu$ in (3.9) by $d_{\xi,\mu}$, where ξ is the number of the row and μ — that of the column. Let us write the principal determinant of the first n equations of n variables (3.9):

$$\begin{vmatrix} d_{1,1} & d_{1,2} & d_{1,3} & \dots & d_{1,n} \\ d_{2,1} & d_{2,2} & d_{2,3} & \dots & d_{2,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{n,1} & d_{n,2} & d_{n,3} & \dots & d_{n,n} \end{vmatrix} = \mathrm{Det}(d_{\xi,\mu}). \tag{3.10}$$

It can be shown that the determinant (3.10) is normal, that is there is a limit

$$\lim_{n\to\infty} \mathrm{Det}\,(d_{\xi,\mu}). \tag{3.11}$$

The wavelengths a_j and the angular frequencies in which we are interested will be determined from the relations (2.9) and (3.10)

$$W(a,\omega) = 0, \tag{3.12}$$

$$\mathrm{Det}\,(d_{\xi,\mu}) = 0. \tag{3.13}$$

Expanding the functions $W(a,\omega)$, Det $(d_{\xi,\mu})$ in power series in a, we obtain

$$\begin{cases} W(a,\omega) = \sum\limits_{\nu=1}^{\infty} \bar{s}_\nu(\omega)a^\nu, \\ \mathrm{Det}\,(d_{\xi,\mu}) = \sum\limits_{\nu=1}^{\infty} \bar{\bar{s}}_\nu(\omega)a^\nu, \end{cases} \tag{3.14}$$

or, taking a finite number of terms β_1, β_2 in the expressions (3.14), we have

$$\begin{cases} W(a,\omega) \approx W_1(a,\omega) = \sum\limits_{\nu=1}^{\beta_1} \bar{s}_\nu(\omega)a^\nu = 0, \\ \mathrm{Det}\,(d_{\xi,\mu}) \approx W_2(a_j,\omega) = \sum\limits_{\nu=1}^{\beta_2} \bar{\bar{s}}_\nu(\omega,a_1,\dots a_{\beta_1-1})a_{\beta_1}^\nu = 0 \qquad (j = 1,2,\dots,\beta_1). \end{cases} \tag{3.15}$$

Using the Fermat or Silvester method, [18], the quantities a_j will be eliminated from the system of equations (3.15), thus obtaining a system of equations where each equation contains either a_j or ω:

$$\sum_{\nu=0}^{\beta_3} E_\nu(U)\omega^\nu = 0, \tag{3.16}$$

$$\sum_{\nu=0}^{\beta_4} F_{\nu,j}(U)a_j^\nu = 0. \tag{3.17}$$

For an analysis of the influence of each particular structural parameter $R,h,l,$ $\varrho,\varrho_s,\lambda_t,\mu_t$ etc. on the possibility of appearance of flutter the Routh, Hurwitz or Michailov method can be used [21], [20], [19]. These methods, which consist in determining the sign of $\mathrm{Re}\,\omega$ [in our case we have $\mathrm{Im}\,\omega > 0$ in view of (2.1)] would require tedious computations. Therefore, to determine the region of appearance of flutter depending on mechanical and aerodynamic parameters the generalized Neumark-Nyquist method will be used (See Sec. 4).

It should be observed that the application of the Hurwitz criterion to the Eq. (3.16), which has complex coefficients $E_\nu(U)$, requires the Eq. (3.16) to be replaced with an equation of the order $2\,\beta_3$ with real coefficients. This equation is obtained on the basis of (3.16) by multiplying it by an equation in which the coefficients $E_\nu(U)$ are replaced with the conjugate coefficients $\overline{E}_\nu(U)$ (cf. [23]). In the latter case, in order to avoid cumbersome computation, the Schur criterion for polynomials with complex coefficients, [24], analogous to that of Hurwitz, can be applied.

4. Dynamic Stability Regions

An analysis of the dynamic stability regions depending on the aerodynamic and mechanical parameters of shells consists in determining the boundary separating the region in which s roots of (3.16) have negative real parts and from that where $\beta_3 - s$ roots have positive real parts, $s = 1, 2, \ldots, \beta_3$, (cf. [13], [21]).
To this end we insert $\omega = i\overline{\omega}$ in (3.16), where the real number $\overline{\omega}$ varies within the limits $-\infty \leqslant \overline{\omega} \leqslant \infty$, and obtain by virtue of (3.16):

$$\sum_{\nu=0}^{\beta_3} E_\nu(\xi,\eta)\,(i\overline{\omega})^\nu = \overline{E}(\xi,\eta,\omega) + i\overline{\overline{E}}(\xi,\eta,\overline{\omega}) = 0, \tag{4.1}$$

where ξ,η are arbitrary parameters of which the influence on the flutter range is the object of our investigation ($\xi = U$, $\eta = l$, for instance).

From the relation (4.1) we have the system of equations

$$\overline{E}(\xi,\eta,\overline{\omega}) = 0, \quad \overline{\overline{E}}(\xi,\eta,\overline{\omega}) = 0. \tag{4.2}$$

Eliminating ω by means of the Fermat method, we obtain:

$$H(\xi,\eta) = 0 \tag{4.3}$$

and, for instance,

$$G(\xi,\overline{\omega}) = 0. \tag{4.4}$$

On the basis of (4.3), (4.4), and computing numerically $\xi,\eta,\overline{\omega}$, we obtain graphically the boundary separating the regions where s roots of (3.16) have $\mathrm{Re}\,\omega < 0$.

5. The Displacement Functions u,v,w, and the Flow Potential Φ

On the basis of the relations (2.1), (3.8), (3.2), the displacement of the middle surface u, v, w of the shell and the perturbation potential of the supersonic flow Φ are expressed in the form

$$\left\{\begin{aligned} u &= \sum_{\nu=1}^{\infty} \tilde{C}_\nu \frac{f(a_\nu)}{\nu!} e^{i(a_\nu x+n\varphi+\omega t)}, \\ v &= \sum_{\nu=1}^{\infty} \tilde{C}_\nu \frac{g(a_\nu)}{\nu!} e^{i(a_\nu x+n\varphi+\omega t)}, \\ w &= \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \tilde{C}_\nu e^{i(a_\nu x+n\varphi+\omega t)}, \\ \Phi &= \sum_{\nu=1}^{\infty} \frac{(\omega+a_\nu U)\varrho}{\nu! m_\nu I_n'(im_\nu R)} I_n(im_\nu r) e^{i(a_\nu x+n\varphi+\omega t)}. \end{aligned}\right. \tag{5.1}$$

The constants $\tilde{C}_\nu$ will be obtained from the normal system of equations (3.9) which has, as is known, ε independent solutions (ε is bounded and positive). Similarly to the case (3.9) it can be shown that the non-homogeneous system of equations obtained from (3.9) is fully regular, therefore it has a unique set of solutions functions (assuming that $\tilde{C}_\varepsilon$ is bounded). From the above it follows that all the series (5.1) are convergent.

References

[1] Р. Д. Степанов, *О флаттере цилиндрических оболочек и панелей, движущихся в потоке газа*, Прикл. Мат. Мех., 2, **25** (1957).

[2] В. В. Болотин, *Колебания и устойчивость упругой цилиндрической оболочки в потоке сжимаемой жидкости*, Инж. Сборник, **25** (1956).

[3] J. Miles, *Supersonic flutter of cylindrical shell*, J. Aero. Sci., 2, **24** (1957); 5, **25** (1958).

[4] R. Leonard, J. Hedgepeth, *On panel flutter and divergence of infinitely long unstiffened and ringstiffened thin walled circular cylinder*, NACA Rep. 1302, 1957.

[5] Z. Dżygadło, S. Kaliski, *Drgania samowzbudne wzmocnionej niesprężystej powłoki cylindrycznej w zlinearyzowanym opływie naddźwiękowym*, Biul. WAT No. XLV, 1959.

[6] Z. Dżygadło, S. Kaliski, *Drgania samowzbudne wzmocnionej ortotropowej, niesprężonej powłoki cylindrycznej w zlinearyzowanym opływie naddźwiękowym*, Biul. WAT No XLV 1959.

[7] А. А. Мовчан, *О влиянии аэродинамического демпфирования на сверхзвуковый флаттер обшивки*, И.А.Н. Мех. и Мат., **1** (1960).

[8] M. Holt, S. Strack, *Supersonic panel flutter of cylindrical shell of finite length*, I.A.S Paper No 60-62, New York, January 25-27, 1960.

[9] В. В. Власов, *Общая теория оболочек*, Москва 1949.

[10] W. Flügge, *Static und Dynamik der Schalen*, Berlin 1957.

[11] А. Маркушевич, *Теория аналитических функций*, Москва 1950.

[12] М. Фихтенгольц, *Курс дифференциального и интегрального исчисления*, Москва 1959.

[13] M. Nyquist, *Regeneration theory*, Bell System Techn. Journ., No 11, 1932.

[14] В. М. Даревский, ред., *Прочность цилиндрических оболочек*, Сборник статей, Москва 1959.

[15] А. Л. Гольденвейзер, *Теория упругих тонких оболочек*, Москва 1953.

[16] В. В. Новожилов, *Теория тонких оболочек*, Москва 1951.

[17] S. Timoschenko. *Theory of plates and shells*, New York 1940.

[18] W. Sierpiński, *Zasady algebry wyższej*, Warszawa 1946.

[19] А. В. Михайлов, *Метод гармонического анализа в теории регулирования*, Автом. Телемех., 4, **3** (1938).

[20] J. Routh, *Advanced part of the dynamics of system of rigid bodies*, vol. 2, London 1930.

[21] A. Hurwitz, *Über die Bedingungen, unter welchen eine Gleichung nur Wurzeln mit negativen reellen Teilen besitzt*, Math. Annal, **46** (1895).

[22] Ю. И. Неймар, *Об определении значений параметров при которых система устойчива*, Автом. Телем., 3, **9** (1948).

[23] В. Л. Залускин, *Справочник по численным методам решения уравнений*, Москва 1960.

[24] J. Schur, *Über algebraische Gleichungen die nur Wurzeln mit negativen Realteilen besitzen*, Zeitschr. Math. Mech., 1921, 307-311.

Streszczenie

DRGANIA POWŁOKI O SKOŃCZONEJ DŁUGOŚCI PRZY NADDŹWIĘKOWYM PRZEPŁYWIE WEWNĘTRZNYM

W pracy autorzy rozważają problem drgań samowzbudzanych cienkiej powłoki cylindrycznej o skończonej długości przy naddźwiękowym przepływie wewnętrznym. Zagadnienie jest rozwiązane przy warunkach brzegowych dla najczęściej spotykanych w praktyce odpowiednich urządzeń: jeden koniec zamocowany, a drugi swobodny (przy innych warunkach brzegowych zagadnienie można rozwiązać analogicznie).

Równania powłoki są wyprowadzone według teorii zagięciowej z uwzględnieniem tłumienia wewnętrznego zgodnie z modelem Voigta analogicznie jak w pracy [5]. Zachowane są jednak pewne człony w równaniach równowagi podobnie jak w [10].

Przepływ wewnętrzny jest rozważany w oparciu o zlinearyzowaną teorię nieustalonego przepływu. Metody podane w tej pracy oparte na sprowadzeniu zagadnienia do nieskończonego układu równań algebraicznych pozwalają na efektywne, numeryczne przeprowadzenie obliczeń (między innymi granic stateczności dynamicznej) i na ewentualne zastosowanie maszyn matematycznych.

Резюме

КОЛЕБАНИЯ ОБОЛОЧКИ КОНЕЧНОЙ ДЛИНЫ ПРИ СВЕРХЗВУКОВОМ ВНУТРЕННЕМ ТЕЧЕНИИ

Рассматриваются самовозбуждающие колебания тонкой цилиндрической оболочки конечной длины со сверхзвуковом внутреннем течениеи. Задача решается при краевых условиях для наиболее часто встречающихся, на практике, соответствующих устройствах, т.е. когда один конец защемлен,

а другой свободный (при других краевых условиях задачу можно решить аналогично).

Уравнения оболочки выводятся согласно теории изгиба с учетом внутреннего демпфирования, согласно модели Фойгта, аналогично работе [5]. Сохраняются однако некоторые члены в уравнениях равновесия подобно как в работе [10].

Внутреннее течение рассматривается, основываясь на линеаризованной теории нестационарного течения.

Методы, приведенные в этой работе, состоят в сведении задачи к бесконечной системе алгебраических уравнений, позволяют на ефективное, численное проведение расчетов (между прочим пределов динамической устойчивости) и возможное использование математических счетных машин.

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LINEARIZED SUPERSONIC FLOW PAST A VIBRATING SURFACE OF A BODY OF REVOLUTION

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1. Introduction

The aim of this paper is to solve the problem of linearized supersonic potential flow past a body of revolution, whose surface vibrates in a harmonic manner, and to determine the pressure of the gas acting on the vibrating surface. The solutions obtained will be applied in next papers to the investigation of self-excited vibrations of cylindrical and conical shells of finite length.

The potential of a pulsating source in a homogeneous supersonic flow, which is known for very long and applied to many cases of flow, [1], [2], [3], can also be used as a basis for the construction of an axially symmetric flow potential past the vibrating surface of a body of revolution.

Considering such sources distributed along the axis of the body W. H. DORRANCE, [4], and N. F. KRASNOW, [5], obtained also the potential of distributed dipoles proportional to $\cos\alpha$ (where α is the angular coordinate of the cylindrical reference frame in which the problem is considered), and applied it to the analysis of the flow past a vibrating rigid body of revolution. In the general case, the form of the potential of the unsteady supersonic flow past a pointed slender body was given in the first approximation by J. W. MILES, [3], valid only in a sufficiently small neighbourhood of the surface of the body. Due to the simplifications made, the only case considered with this approximation is that of perturbed transverse flow treated as incompressible, the axial variable and the time being treated as parameters.

The approximate form of the potential of a supersonic flow past a vibrating body of revolution of a considerable length was determined by G. I. KOPZON, [6]. The usefulness of the approximation of this seems to be somewhat doubtful if it is borne in mind that (as follows from the Eq. (9) of [6]) the rejected part tends to infinity as $1/y^2$, if $y \to 0$, that is if the point on the surface of the body tends to the front edge and the remaining approximate expression for the potential is bounded in the neighbourhood of the point $y = 0$.

The Laplace transform of the potential of the supersonic flow past a vibrating cylinder has been given recently by M. Holt and S. L. Strack, [7]. However, the effective determination of the original was not considered.

In the present paper, the general form is determined of the potential of the external supersonic flow past the vibrating surface of a body of revolution. Making use of the potential obtained, the solution of the problem of flow past a vibrating surface of revolution is reduced in the general case to the solution of a certain Volterra integral equation of the second kind. For a vibrating cylinder, the relation between the pressure and the normal component of the displacement is derived in the form of an asymptotic expansion in ascending powers of the inverse Mach number. From the expression thus obtained, we obtain, in the first approximation, an equation for the pressure in agreement with the so-called piston approximation. In contrast to the plane flow, where the error of the piston approximation is of the order $1/M^2$ in relation to 1 (M is the Mach number of the unperturbed flow), the error in the case of a cylinder will be of the order $1/M$. In this connection, if self-excited vibrations of cylindrical shells are considered, further terms of the expansion should be used for greater accuracy, with moderate Mach numbers ($M < 10$).

For pointed slender bodies the second linear approximation of the potential is given. It has an error of order double that of the first approximation, [3], takes into consideration the compressibility of the flow, and enables a more accurate description of the phase angle between the normal component of the displacement of points of the vibrating surface and the pressure, which play an essential role in the investigation of self-excited vibrations. The potential derived is used for the determination of the pressure on a vibrating cone.

2. Fundamental Equations

Let us consider an external supersonic flow past the vibrating surface of a body of revolution the OX-axis of which is parallel to the unperturbed flow of velocity $U_0 > a_0$ (Fig. 1).

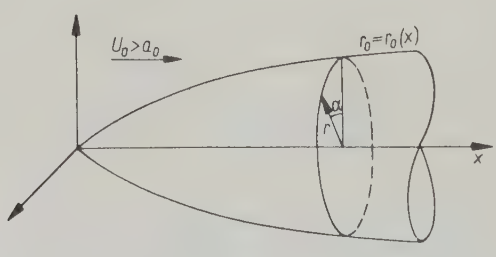

Fig. 1

The form of the body the surface of which is determined at rest, by the equation $r_0 = r_0(x)$, and the Mach number M of the unperturbed flow satisfy the conditions

$$\left(\frac{dr_0}{dx}\right)^2 \ll 1, \quad \sqrt{M^2-1}\,\frac{dr_0}{dx} < \delta < 1. \tag{2.1}$$

Assuming that the amplitudes of vibration of the points of the surface of the body are sufficiently small and bearing in mind (2.1) (if the number δ is sufficiently less than 1), the problem may be solved on the grounds of the theory of linearized supersonic potential flow.

Let us introduce dimensionless quantities. The coordinates x, r and the normal component of the displacement of the surface of the body W will be referred to L, the time t to the quotient L/U_0 and the perturbing potential Φ to the product U_0L. For the reference lenght L, we shall choose the maximum radius of the body or its length depending on the necessity.

The dimensionless perturbing potential $\Phi(x, r, \alpha; t)$ satisfies the equation:

$$\mu^2\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial r^2}-\frac{1}{r}\frac{\partial\Phi}{\partial r}-\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\alpha^2}+2M^2\frac{\partial^2\Phi}{\partial x\partial t}+M^2\frac{\partial^2\Phi}{\partial t^2}=0, \tag{2.2}$$

where $\mu=\sqrt{M^2-1}$.

The boundary condition of the Eq. (2.2) coupling the vibration of the surface with the flow has, bearing in mind (2.1), the form:

$$\left[\frac{\partial\Phi}{\partial r}-\frac{dr_0}{dx}\frac{\partial\Phi}{\partial x}\right]_{r=r_0(x)}=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)\left[1+\frac{1}{2}\left(\frac{dr_0}{dx}\right)^2\right]W(x,\alpha;t). \tag{2.3}$$

In view of the supersonic character of the flow, the potential Φ must also satisfy the condition

$$\text{for } x\leqslant\mu r, \quad \Phi(x,r,\alpha;t)\equiv 0. \tag{2.4}$$

The potential Φ being thus obtained, the difference ΔP between the pressure P_0 in the unperturbed flow and the pressure P in the perturbed flow will be determined from the equation

$$\Delta P=P_0-P=\varrho_0U_0^2\left(\frac{\partial\Phi}{\partial t}+\frac{\partial\Phi}{\partial x}\right), \tag{2.5}$$

where ϱ_0 is the gas density in the unperturbed flow.

In what follows we shall be concerned with the analysis of the vibration

$$W(x,\alpha;t)=\cos n\alpha\, e^{i\omega t}w_n(x). \tag{2.6}$$

In the connection the potential will be sought for in the form

$$\Phi(x,r,\alpha;t)=\cos n\alpha\exp[i(\omega t-\beta x)]\varphi_n(x,z), \tag{2.7}$$

where $z=\mu r$, $\beta=M^2\omega/M^2-1$, $n=0,1,2,\ldots$, and the frequency of vibration ω may, in general, be complex.

Knowledge of the solutions of the form (2.7) makes possible to study vibration of the form (2.6) and, solutions of more complicated cases may, by adding such expressions, be obtained.

Substituting (2.6) and (2.7) in (2.2) to (2.4), and denoting

$$z_0(x)=\mu r_0(x),\quad \gamma=\frac{M\omega}{M^2-1},\quad w_n^*(x)=\frac{1}{\mu}e^{i\beta x}\left(\frac{d}{dx}+i\omega\right)\left[1+\frac{1}{2}\left(\frac{dr_0}{dx}\right)^2\right]w_n(x),$$

the flow problem under consideration reduces to

$$\frac{\partial^2\varphi_n}{\partial x^2}-\frac{\partial^2\varphi_n}{\partial z^2}-\frac{1}{z}\frac{\partial\varphi_n}{\partial z}+\left(\gamma^2+\frac{n^2}{z^2}\right)\varphi_n=0, \tag{2.8}$$

$$\left[\frac{\partial\varphi_n}{\partial z}-\frac{1}{\mu}\frac{dr_0}{dx}\frac{\partial\varphi_n}{\partial x}+\frac{i\beta}{\mu}\frac{dr_0}{dx}\varphi_n\right]_{z=z_0(x)}=w_n^*(x), \tag{2.9}$$

$$\text{for } x\leqslant z \quad \varphi_n(x,z)\equiv 0. \tag{2.10}$$

The pressure difference determined by (2.5) may now be written in the form

$$\Delta P(x,\alpha,z;t)=\varrho_0 U_0^2\cos n\alpha\, e^{i\omega t}\Delta p_n(x,z), \tag{2.11}$$

where the function $\Delta p_n(x,z)$ is expressed thus:

$$\Delta p_n(x,z)=e^{-i\beta x}\left(\frac{\partial\varphi_n}{\partial x}-i\frac{\omega}{\mu^2}\varphi_n\right). \tag{2.12}$$

3. Solution of the Eq. (2.8)

Let us proceed now to find the solution of (2.8) satisfying the condition (2.10). We apply the Laplace transformation to (2.8) in relation to x, and denote

$$\mathcal{L}\varphi_n(x,z)=\int_0^\infty e^{-sx}\varphi_n(x,z)dx=\overline{\varphi}_n(s,z).$$

Assuming that $\varphi_n(x, z)$ is continuous for $x>0$ and $z>0$, and satisfies (2.10) and (2.8), we obtain:

$$\frac{d^2\overline{\varphi}_n}{dz^2}+\frac{1}{z}\frac{d\overline{\varphi}_n}{dz}-\left(s^2+\gamma^2+\frac{n^2}{z^2}\right)\overline{\varphi}_n=0. \tag{3.1}$$

The solution of the Eq. (3.1), the original function of which satisfies the conditions (2.10), has the form

$$\overline{\varphi}_n(s,z)=\overline{A}_n(s)K_n(z\sqrt{s^2+\gamma^2}), \tag{3.2}$$

where $\overline{A}_n(s)$ — is an arbitrary function, K_n — a modified Bessel function of the second kind, the asymptotic appraisal of which for $z\to\infty$ is

$$K_n(z)=\sqrt{\frac{\pi}{2z}}\,e^{-z}\left[1+0\left(\frac{1}{z}\right)\right].$$

The solution of (2.8), satisfying the condition (2.10), will be obtained by treating (3.2) as a convolution and writing its original function in the form:

$$\varphi_n(x,z)=\int_0^{x-z}A_n(\xi)N_n(x-\xi,z)d\xi, \tag{3.3}$$

where

$$N_n(x,z)=\mathcal{L}^{-1}K_n(z\sqrt{s^2+\gamma^2}).$$

The kernel $N_n(x, z)$ of the expression (3.3) will be determined by applying the Efros theorem

$$\left\{\begin{array}{ll} \text{for } x<z & N_n(x,z) = 0, \\ \text{for } x>z & N_n(x,z) = \dfrac{\operatorname{ch}\left(n \operatorname{arch}\dfrac{x}{z}\right)}{\sqrt{x^2-z^2}} - \gamma x \displaystyle\int_z^x J_1(\gamma\sqrt{x^2-\xi^2})\times \end{array}\right. \tag{3.4}$$

$$\times \frac{\operatorname{ch}\left(n \operatorname{arch}\dfrac{\xi}{z}\right)}{\sqrt{\xi^2-z^2}\sqrt{x^2-\xi^2}} d\xi + \gamma \int_z^x \frac{\operatorname{ch}\left(n \operatorname{arch}\dfrac{\xi}{z}\right)}{\sqrt{\xi^2-z^2}} d\xi \int_\xi^x J_0(\gamma\sqrt{\eta^2-\xi^2}) J_1[\gamma(x-\eta)] \frac{d\eta}{x-\eta},$$

where J_0 and J_1 are Bessel functions of the first kind. For $n = 0$, the Eq. (3.4) may be transformed to give:

$$N_0(x, z) = \begin{cases} 0 & \text{for } x<z, \\ \dfrac{\cos(\gamma\sqrt{x^2-z^2})}{\sqrt{x^2-z^2}} & \text{for } x>z. \end{cases} \tag{3.5}$$

Substituting (3.5) in (3.3), we obtain:

$$\varphi_0(x, z) = \int_0^{x-z} A_0(\xi) \frac{\cos(\gamma\sqrt{(x-\xi)^2-z^2})}{\sqrt{(x-\xi)^2-z^2}} d\xi. \tag{3.6}$$

Now, after substituting (3.6) in (2.7), we obtain the known potential of linearly distributed pulsating sources in a homogeneous supersonic flow [1].

In the case of $n > 0$ we can obtain a solution of the Eq. (2.8) in the form (3.3) the kernel of which is of a much simpler form than that given by the Eq. (3.4). For this purpose, the following method will be used.

There is a well known property of the Eq. (2.8), used in the theory of flow past bodies of revolution, consisting in that its solution for $n = 1$ can be obtained by differentiating with respect to z, the solution for $n = 0$.

This property may be generalized to the case of any $n = 1, 2, 3, \ldots$ For,

$$l_n\left(\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} - \frac{(n-1)^2}{z^2}\right) = \left(\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} - \frac{n^2}{z^2}\right) l_n, \tag{3.7}$$

if

$$l_n = z^{n-1}\frac{\partial}{\partial z} z^{1-n}. \tag{3.7.1}$$

Applying n times the Eq. (3.7) we obtain

$$L_n\left(\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z}\right) = \left(\frac{\partial^2}{\partial z^2} + \frac{1}{z}\frac{\partial}{\partial z} - \frac{n^2}{z^2}\right) L_n, \tag{3.8}$$

where

$$L_n = \prod_{k=n}^{1} l_k = z^{n-1}\frac{\partial}{\partial z}\frac{1}{z}\frac{\partial}{\partial z}\cdots\frac{1}{z}\frac{\partial}{\partial z} = z^n\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n. \tag{3.8.1}$$

From (3.8) it follows that if a sufficiently regular solution of (2.8) is known for $n = 0$, we can obtain the solution for $n = 1, 2, 3, \ldots$ in the form:

$$\varphi_n(x, z) = L_n\varphi_0(x, z). \tag{3.9}$$

Let us use (3.9) to determine a simpler form of the kernel of the potential $\varphi_n(x,z)$ than that given by (3.4). Let us assume (3.6) for the initial potential $\varphi_0(x, z)$. In view of the necessity of manifold differentiation we shall write it in the form

$$\varphi_0(x, z) = \int_0^{\operatorname{arch}\frac{x}{z}} A_0(x - z\operatorname{ch}u)\cos(\gamma z\operatorname{sh}u)\,du \tag{3.10}$$

obtained from (3.6) after changing the integration variable $\xi = x - z\operatorname{ch}u$. Substituting (3.10), in (3.9) and assuming that $A_0(\xi)$ is differentiable a sufficient number of times and $A_0(0) = A_0'(0) = \ldots = A_0^{(n-1)}(0) = 0$, we obtain:

$$\varphi_n(x, z) = L_n\varphi_0(x, z) = \sum_{k=0}^{n}\int_0^{\operatorname{arch}\frac{x}{z}} A_0^{(k)}(x - z\operatorname{ch}u)\,[B_{nk}\cos(\gamma z\operatorname{sh}u)\operatorname{ch}nu + C_{nk}\sin(\gamma z\operatorname{sh}u)\operatorname{sh}nu]\,du, \tag{3.11}$$

where B_{nk} and C_{nk} are constants.

Each term of the right-hand member of (3.11) identically satisfies (2.8). Therefore from (3.11), two different expressions for $\varphi_n(x, z)$ are obtained:

$$\varphi_n(x, z) = \int_0^{\operatorname{arch}\frac{x}{z}} q_{1n}(x - z\operatorname{ch}u)\cos(\gamma z\operatorname{sh}u)\operatorname{ch}nu\,du = \tag{3.12}$$

$$= \frac{1}{z^n}\int_0^{x-z} q_{1n}(\xi)\frac{\cos\left[\gamma\sqrt{(x-\xi)^2 - z^2}\right]}{\sqrt{(x-\xi)^2 - z^2}} f_{1n}(x - \xi, z)\,d\xi \qquad \text{for } n = 0, 1, 2, \ldots$$

$$\varphi_n(x, z) = \int_0^{\operatorname{arch}\frac{x}{z}} q_{2n}(x - z\operatorname{ch}u)\sin(\gamma z\operatorname{sh}u)\operatorname{sh}nu\,du = \tag{3.13}$$

$$= \frac{1}{z^n}\int_0^{x-z} q_{2n}(\xi)\frac{\sin\left[\gamma\sqrt{(x-\xi)^2 - z^2}\right]}{\sqrt{(x-\xi)^2 - z^2}} f_{2n}(x, -\xi,)z)\,d\xi \qquad \text{for } n = 1, 2, 3, \ldots,$$

where

$$f_{1n}(x, z) = \frac{1}{2}\left[(x + \sqrt{x^2 - z^2})^n + (x - \sqrt{x^2 - z^2})^n\right],$$

$$f_{2n}(x, z) = \frac{1}{2}\left[(x + \sqrt{x^2 - z^2})^n - (x - \sqrt{x^2 - z^2})^n\right],$$

$q_{1n}(\xi)$ and $q_{2n}(\xi)$ are arbitrary functions.

We can also construct the potential $\varphi_0(x, z)$, completing (3.13) for $n = 0$. It has the form

$$\varphi_0(x, z) = \int_0^{\operatorname{arch}\frac{x}{z}} q_{20}(x - z\operatorname{ch} u)\sin(\gamma z \operatorname{sh} u)u\,du = \tag{3.13.1}$$

$$= \int_0^{x-z} q_{20}(\xi)\frac{\sin[\gamma\sqrt{(x-\xi)^2 - z^2}]}{\sqrt{(x-\xi)^2 - z^2}}\ln\left[\frac{x-\xi}{z} + \sqrt{\frac{(x-\xi)^2}{z^2} - 1}\right]d\xi.$$

The two forms (3.12) and (3.13) of the potential are equivalent. In what follows, one form or other will be used as required.

For the applications, it will be necessary to know the Laplace transform of the potential (3.12). In view of the uniqueness of the problem the transform sought for may differ from (3.2) by a form of the function of the parameter s only. Therefore it can be determined by equating the transform of the asymptotic expression obtained from (3.12) for $z \to 0$ with the corresponding asymptotic equation obtained from (3.2). In this way, we obtain for $z \to 0$ and $n = 1, 2, 3, \ldots$

$$\frac{1}{z^n}\int_0^{x-z} q_{1n}(\xi)\frac{\cos[\gamma\sqrt{(x-\xi)^2 - z^2}]}{\sqrt{(x-\xi)^2 - z^2}} f_{1n}(x-\xi, z)\,d\xi \approx$$

$$\approx \frac{2^{n-1}}{z^n}\int_0^x q_{1n}(\xi)(x-\xi)^{n-1}\cos[\gamma(x-\xi)]\,d\xi,$$

$$\mathcal{L}\,\frac{2^{n-1}}{z^n}\int_0^x q_{1n}(\xi)(x-\xi)^{n-1}\cos[\gamma(x-\xi)]\,d\xi = \frac{2^{n-1}(n-1)!}{z^n}\,\bar{q}_{1n}(s)\,\frac{\operatorname{Re}'(s+i\gamma)^n}{(s^2+\gamma^2)^n}$$

and

$$\bar{A}_n(s)K_n(z\sqrt{s^2+\gamma^2}) \approx \frac{2^{n-1}(n-1)!}{z^n}\,\frac{\bar{A}_n(s)}{(s^2+\gamma^2)^{n/2}}.$$

Hence

$$\bar{A}_n(s) = \bar{q}_{1n}(s)\frac{\operatorname{Re}'(s+i\gamma)^n}{(s^2+\gamma^2)^{n/2}}$$

and, finally,

$$\mathcal{L}\,\frac{1}{z^n}\int_0^{x-z} q_{1n}(\xi)\frac{\cos[\gamma\sqrt{(x-\xi)^2 - z^2}]}{\sqrt{(x-\xi)^2 - z^2}} f_{1n}(x-\xi, z)\,d\xi = \tag{3.14}$$

$$= \bar{q}_{1n}(s)\frac{\operatorname{Re}'(s+i\gamma)^n}{(s^2+\gamma^2)^{n/2}}K_n(z\sqrt{s^2+\gamma^2}),$$

where Re' denotes the real part assuming that s is real.

4. Solution of the Flow Problem

We shall discuss now a method for solving the problem of external flow. This consists in reducing the problem to the solution of a Volterra integral equation of the second kind.

The solution may be obtained by using the potential $\varphi_n(x, z)$ directly in the form (3.13). Making use of (3.12), we can also construct a potential with the kernel $R_n(x, z)$, bounded and non-zero for $x = z$ and use it to solve the problem.

In the first case, substituting (3.13) in the boundary condition (2.9), we obtain the equation

$$\int_0^{x-z_0(x)} \frac{M_n[x-\xi, z_0(x)]}{\sqrt{(x-\xi)^2-z_0^2(x)}} q_{2n}(\xi)d\xi = -[z_0(x)]^{n+1} w_n^*(x), \tag{4.1}$$

where $M_n(x, z)$ is a function bounded and non-zero for $x = z$;

$$M_0(x, z) = \frac{\sin(\gamma\sqrt{x^2-z^2})}{\sqrt{x^2-z^2}} \left\{x + \frac{z}{\mu}\frac{dr_0(x)}{dx}\left[1 - i\beta\sqrt{x^2-z^2}\ln\left(\frac{x}{z}+\sqrt{\frac{x^2}{z^2}-1}\right)\right]\right\} +$$

$$+\left[\gamma\cos(\gamma\sqrt{x^2-z^2}) - \frac{\sin(\gamma\sqrt{x^2-z^2})}{\sqrt{x^2-z^2}}\right]\left(z+\frac{x}{\mu}\frac{dr_0}{dx}\right)\frac{\ln\left(\frac{x}{z}+\sqrt{\frac{x^2}{z^2}-1}\right)}{\sqrt{\frac{x^2}{z^2}-1}},$$

$$M_n(x, z) = \frac{\sin(\gamma\sqrt{x^2-z^2})}{\sqrt{x^2-z^2}}\left[n f_{1n}(x, z)\left(x+\frac{z}{\mu}\frac{dr_0}{dx}\right) - \frac{i\beta}{\mu}\frac{dr_0}{dx} z\sqrt{x^2-z^2} f_{2n}(x, z)\right] +$$

$$+\left[\gamma\cos(\gamma\sqrt{x^2-z^2}) - \frac{\sin(\gamma\sqrt{x^2-z^2})}{\sqrt{x^2-z^2}}\right]\left(z+\frac{x}{\mu}\frac{dr_0}{dx}\right)\frac{f_{2n}(x, z)}{\sqrt{\frac{x^2}{z^2}-1}}, \qquad n = 1, 2, 3, \ldots .$$

Substituting $x_1 = x - z_0(x)$ for x in (4.1), and applying the operator $\int_0^y \frac{dx_1}{\sqrt{y-x_1}}$, we obtain a Volterra equation of the first kind, which becomes, on differentiating, the sought-for equation of the second kind.

The above method is convenient for bodies of simpler forms (cylinders, cones). For more composite forms another method is better suited. It consists in applying a potential with the kernel bounded for $z > 0$ and non-zero for $z = x$.

Such a potential,

$$\varphi_n(x, z) = \int_0^{x-z} g_n(\xi) R_n(x-\xi, z) d\xi \tag{4.2}$$

can be constructed by means of (3.12). Indeed, assuming $q_{1n}(\xi) = 1/\sqrt{\xi}$, we obtain:

(4.3) $$R_n(x, z) = \frac{1}{z^n}\int_0^{x-z} \frac{\cos[\gamma\sqrt{(x-\xi)^2-z^2}]}{\sqrt{\xi[(x-\xi)^2-z^2]}} f_{1n}(x-\xi, z)d\xi.$$

The function $R_n(x, z)$ is bounded for $z > 0$ and non-zero for $x = z$, as can be seen by introducing in (4.3) a new integration variable by means of the relation $\xi = (x-z)\sigma$.

(4.4) $$R_n(x, z) = \frac{1}{z^n}\int_0^1 \frac{\cos\{\gamma\sqrt{(x-z)(1-\sigma)[x(1-\sigma)+z(1+\sigma)]}\}}{\sqrt{\sigma(1-\sigma)}\sqrt{x(1-\sigma)+z(1+\sigma)}} f_{1n}[x(1-\sigma)+z\sigma, z]d\sigma,$$

(4.5) $$R_n(z, z) = \frac{1}{\sqrt{2z}}\int_0^1 \frac{d\sigma}{\sqrt{\sigma(1-\sigma)}} = \frac{\pi}{\sqrt{2z}} \quad \text{for} \quad x \to z.$$

Substituting (4.2) with the kernel (4.3) in the boundary condition (2.9), and bearing in mind (4.5), we obtain directly the Volterra equation of the second kind:

(4.6) $$g_n(x_1) - \int_0^{x_1} g_n(\xi)H_n(x_1, \xi)d\xi = h_n(x_1),$$

where

$$h_n(x_1) = -\frac{\sqrt{2z_0[x(x_1)]}}{\pi\left(1+\frac{1}{\mu}\frac{dr_0[x(x_1)]}{dx}\right)} w_n^*[x(x_1)],$$

$$H_n(x_1, \xi) = \left\{\frac{\sqrt{2z}}{\pi\left(1+\frac{1}{\mu}\frac{dr_0(x)}{dx}\right)}\left[\frac{\partial R_n(x-\xi, z)}{\partial z} - \frac{1}{\mu}\frac{dr_0(x)}{dx}\frac{\partial R_n(x-\xi, z)}{\partial x} + \right.\right.$$
$$\left.\left. + \frac{i\beta}{\mu}\frac{dr_0(x)}{dx}R_n(x-\xi, z)\right]\right\}_{\substack{z=z_0[x(x_1)] \\ x=x(x_1)}}$$

and $x(x_1)$ is the inverse function of $x_1 = x - z_0(x)$.

The solution of (4.6) will be obtained in the form:

(4.7) $$g_n(x_1) = h_n(x_1) + \int_0^{x_1} H_n^*(x_1, \xi)h_n(\xi)d\xi,$$

where

$$H_n^*(x_1, \xi) = H_n(x_1, \xi) + \sum_{m=1}^{\infty} H_{nm}(x_1, \xi)$$

and

$$H_{nm}(x_1, \xi) = \int_\xi^{x_1} H_n(x_1, \tau)H_{nm-1}(\tau, \xi)d\tau.$$

5. Determination of the Pressure Acting on a Vibrating Cylinder in a Supersonic Flow

In the case of a cylinder we can, without solving equations of the form (4.1) or (4.6), obtain directly, in an explicit form, the relation between the pressure and the normal displacement component of points on the surface of the cylinder.

Let us assume for the case now under consideration that the reference length is $L = R$, where R is the radius of the cylinder, and that the vibrating cylinder constitutes a portion of a body of revolution satisfying the conditions (2.1). The reference frame will be assumed according to Fig. 2.

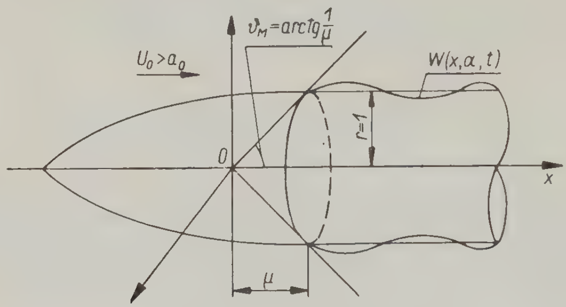

Fig. 2

Let us apply to (2.12) the Laplace transformation in relation to x. In the relation thus obtained we substitute $\bar{\varphi}_n(s, z)$ in the form (3.2). Then, we obtain

$$\overline{\Delta p}_n(s, z) = (s+i\omega)\bar{A}_n(s+i\beta)K_n[z\sqrt{(s+i\beta)^2+\gamma^2}\,]. \tag{5.1}$$

Next, assuming that $w_n(x)$ is continuous for $x > 0$ and $w_n(x) \equiv 0$ for $x \leqslant \mu$ (Fig. 2), and bearing in mind that now $z_0(x) = \mu$, we obtain, after transforming the boundary condition (2.9), the relation:

$$\bar{A}_n(s+i\beta) = \frac{(s+i\omega)\bar{w}_n(s)}{\mu\sqrt{(s+i\beta)^2+\gamma^2}\,K_n'(\mu\sqrt{(s+i\beta)^2+\gamma^2}\,)}. \tag{5.2}$$

Substituting (5.2)) in (5.1) on the surface of the cylinder (for $z = \mu$), we obtain,

$$\Delta p_n(s, \mu) = \frac{(s+i\omega)^2}{\mu\sqrt{(s+i\beta)^2+\gamma^2}}\;\frac{K_n[\mu\sqrt{(s+i\beta)^2+\gamma^2}\,]}{K_n'[\mu\sqrt{(s+i\beta)^2+\gamma^2}\,]}\,\bar{w}_n(s). \tag{5.3}$$

The original of (5.3) can of course be sought-for by making use of the residues at the singular points and the branch points of the above relation. In this way an accurate equation is obtained. However, its form is useful for numerical computation only and is inconvenient for other applications for instance for the analysis of self-excited vibration of cylindrical shells.

A different form of the original, very convenient for applications, may be obtained by making use, for the inverse transformation of (5.3), of the asymptotic expansion of the functions $K_n(z)$ and $K_n'(z)$ for large z. Such a procedure is justified if it is assumed that the number M is sufficiently high. Then, the coefficient $\mu = \sqrt{M^2-1}$ appearing in the argument of K_n and K_n', may be treated as a large

parameter. It is true that this makes impossible the consideration of a certain range of M near to unity. That, however, is no serious limitation on applications of practical importance. The asymptotic expansion $K_n(z)$ has, for large z the form [8]:

$$K_n(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\left[\sum_{k=0}^{m-1}\frac{(n,k)}{(2z)^k}+\theta\frac{(n,m)}{(2z)^m}\right], \tag{5.4}$$

and we must have $m > n-1/2$ and $|\theta| < 1$ if $\operatorname{Re} z \geqslant 0$,

$$(n,0) = 1, \qquad (n,k) = \frac{(4n^2-1)(4n^2-3^2)\dots[4n^2-(2k-1)^2]}{2^{2k}k!} \qquad \text{for } k = 1, 2, 3, \dots$$

From (5.4), we obtain the following expansion of the function $K_n(z)/K_n'(z)$ if $m > n+1/2$

$$\frac{K_n(z)}{K_n'(z)} = -\left[\sum_{k=1}^{m-1}\frac{\overline{(n,k)}}{(2z)^k}+\theta\frac{\overline{(n,m)}}{(2z)^m}\right], \tag{5.5}$$

where $\overline{(n,0)} = 1$, $\overline{(n,1)} = -1$,

$$\overline{(n,k)} = (n,k)-(n,k)'-\sum_{j=1}^{k-1}\overline{(n,j)}(n,k-j)' \qquad \text{for} \qquad k = 2,3,4,\dots,m-1,$$

and

$$\overline{(n,m)} = (n,m)-(n,m)'-\frac{1}{\theta}\sum_{j=1}^{m-1}\overline{(n,j)}(n,m-j)',$$

and

$$(n,k)' = \frac{1}{4k}(4n^2+4k^2-1)(n,k-1).$$

On the basis of (5.5) we can write the asymptotic expansion of (5.3) for high Mach numbers:

$$\overline{\Delta p}_n(s,\mu) = -\frac{(s+i\omega)^2\overline{w}_n(s)}{\mu\sqrt{(s+i\beta)^2+\gamma^2}}\left\{\sum_{k=0}^{m-1}\frac{\overline{(n,k)}}{[2\mu\sqrt{(s+i\beta)^2+\gamma^2}]^k}+\right.$$
$$\left.+\frac{\theta\overline{(n,m)}}{[2\mu\sqrt{(s+i\beta)^2+\gamma^2}]^m}\right\}. \tag{5.6}$$

Performing the inverse transformation of (5.6) term-by-term, we obtain(¹)

$$\Delta p_n(x,\mu) = -\frac{1}{\mu}\left(\frac{d}{dx}+i\omega\right)\int_0^x\left[\frac{dw_n(\xi)}{d\xi}+i\omega w_n(\xi)\right]e^{-i\beta(x-\xi)}\left[\sum_{k=0}^{m-1}\frac{\overline{(n,k)}}{(2\mu)^k}\psi_k(x-\xi)+\right.$$
$$\left.+\theta\frac{\overline{(n,m)}}{(2\mu)^m}\psi_m(x-\xi)\right]d\xi. \tag{5.7}$$

(¹) After the inverse transformation of (5.6), the reference frame is shifted to the right by μ, so that the lower bound of the integral in (5.7) is zero, which is more convenient for the applications of the above relation.

The function $\psi_k(x)$ have the form:

for $k = 2\nu$ $\quad \psi_{2\nu}(x) = \mathcal{L}^{-1}(s^2+\gamma^2)^{-\nu-\frac{1}{2}} = \dfrac{\nu!}{(2\nu)!}\left(\dfrac{2x}{\gamma}\right)^{\nu} J_\nu(\gamma x),$

for $k = 2\nu+1$ $\quad \psi_{2\nu+1}(x) = \mathcal{L}^{-1}(s^2+\gamma^2)^{-\nu-1} = \dfrac{\sqrt{\pi}}{\nu!}\left(\dfrac{x}{2\gamma}\right)^{\nu+\frac{1}{2}} J_{\nu+\frac{1}{2}}(\gamma x),$

where $\nu = 0, 1, 2, 3, \ldots$ and J_ν is the Bessel function of the first kind and the ν-th order.

From the expansion obtained it follows that if the accurate expression for the function $\Delta p_n(x, \mu)$ is replaced by the partial sum of the first m terms of (5.7), the error will always be of an order higher by $1/\mu$ than the last of the retained terms. For $m > n+1/2$ the absolute value of the error will be less than the first of the rejected terms. If $M \to \infty$ the error tends to zero. For a fixed value of M there exists a number m, such that the absolute value of the error is minimum.

Numerical computations have been performed for the quantity

$$\left| \frac{\theta \dfrac{\overline{(n, m)}}{(2\mu)^m}\psi_m(x)}{\displaystyle\sum_{k=0}^{m-1} \dfrac{\overline{(n, k)}}{(2\mu)^k}\psi_k(x)} \right| 100\%$$

characterizing the convergence of the initial part of the expansion (5.7). The results are given in Table 1. The computations have been performed for $n = 0;\ 2;\ 4$, $M = 2;4$, $x = 0{,}5;\ 1;\ 1{,}5;\ 2;\ 3;\ 4$, $m = 5$ and $\omega = 1$. For $\omega > 1$ the rate of convergence is higher and for $\omega < 1$ it decreases slightly.

Table 1

M	x \ n	0	2	4
2	0.5	—	—	0.9%
	1	0.06%	1.1%	49%
	1.5	0.6%	7%	—
	2	3.8%	42%	
4	1	—	—	0.47%
	1.5		—	5.6%
	2	0.035%	0.56%	39%
	3	0.32%	7.5%	—
	4	1.7%	48%	

From the computations and the form of the expansion it follows that for $M > 1{,}5$ the value of the function $\Delta p_n(x, \mu)$ can be determined with a high degree of accuracy by retaining only a few terms (3 to 5) of (5.7) provided that the following inequalities hold:

$$\text{for } n = 0 \quad x/M \leqslant 1$$
$$\text{and for } n = 1, 2, 3, \ldots \quad nx/M \leqslant 1.$$

If the ratio nx/M (for $n \neq 0$) or x/M (for $n = 0$) increases above unity the rate of convergence of the initial part of the expansion decreases. For sufficiently large values of the ratio nx/M or x/M the expansion becomes divergent from the first term inclusive.

The relation (5.7) can be represented in a still simpler form, very convenient for applications. For this purpose, let us observe that assuming for the value of the function $\Delta p_n(x,\mu)$ the partial sum of m terms of (5.7), we obtain Δp_n with

an error of the order $0\,(1/\mu^m)$, which is equivalent to an error of the order $0\,(1/M^m)$, because $\mu = \sqrt{M^2-1}$.

Since the Mach number appears in terms of the retained partial sum, not only in the coefficient μ but also in β and γ, therefore, with no increase of the order of the error, we can obtain a simpler formula for $\Delta p_n(x,\mu)$ if the m-th partial sum (5.7) is expanded in a series in $1/M$, the retained terms being those with coefficients of lower order than $1/M^{m+1}$. Then, we obtain:

$$(5.8)\quad \Delta p_n(x,\mu) = \frac{-1}{M}\left[\frac{dw_n(x)}{dx}+i\omega w_n(x)\right]\sum_{k=0}^{c\left(\frac{m}{2}\right)}\frac{b_k}{M^{2k}}+$$
$$+\frac{1}{M^2}\int_0^x\left[\frac{dw_n(\xi)}{d\xi}+i\omega w_n(\xi)\right]e^{-i\omega(x-\xi)}\sum_{k=0}^{m-2}\frac{1}{M^k}\sum_{j=0}^{k}b_{nj}^k(x-\xi)^j d\xi+0\left(\frac{1}{M^{m+1}}\right),$$

where the symbol $c(m/2)$ denotes the integer part of $m/2$, and b_k and b_{nj}^k—the corresponding constants. For $m \leqslant 5$ they have the form:

$$b_0 = 1,\quad b_1 = \frac{1}{2},\quad b_2 = \frac{3}{8};\quad b_{n0}^0 = \frac{1}{2},\quad b_{n0}^1 = i\omega,\quad b_{n1}^1 = \frac{1}{2}\omega^2+\frac{n^2}{2}-\frac{3}{8};$$

$$b_{n0}^2 = \frac{1}{2},\quad b_{n1}^2 = -i\omega,\quad b_{n2}^2 = -\frac{1}{4}\omega^2-\frac{n^2}{2}+\frac{3}{16};\quad b_{n0}^3 = \frac{3}{2}i\omega,$$

$$b_{n1}^3 = \frac{9}{4}\omega^2+\frac{3}{4}n^2-\frac{9}{16},\quad b_{n2}^3 = -\frac{3}{4}i\omega\left(\omega^2+n^2-\frac{3}{4}\right),$$

$$b_{n3}^3 = -\frac{1}{16}\left[\omega^4+\omega^2\left(2n^2-\frac{3}{2}\right)+n^4-\frac{29}{6}n^2+\frac{63}{48}\right].$$

Let us observe that using the first approximation of the expansion (5.8)

$$(5.9)\qquad \Delta p_{n1}(x,\mu) = \frac{-1}{M}\left[\frac{dw_n(x)}{dx}+i\omega w_n(x)\right],$$

we obtain a formula representing the pressure difference identical with the known equation used for the investigation of the self-excited vibration of plates and shells, which is the linear part of the so-called piston approximation to the pressure difference in a supersonic flow. In contrast to the plane problem, where, using the relation (5.9), the error is of the order $1/M^2$ in relation to unity, in the present case of a cylinder the error is of the order $1/M$, because in the expansion (5.8) there appear all the powers of the inverse M number. Thus, from the form of the Eq. (5.8), it follows that if self-excited vibration of cylindrical shells is investigated, further terms of the expansion should, for increased accuracy, be taken for moderate Mach numbers ($M<10$).

6. Approximate Potential of Flow past the Vibrating Surface of a Pointed Slender Body. The Pressure on a Vibrating Cone

We shall now derive an approximate expression for the potential of supersonic flow past the vibrating surface of a pointed slender body of revolution. This approximation may be called the second linear approximation of the known theory of flow past pointed slender bodies, [3].

Let us assume now the reference length L to be equal to the length of the vibrating part of the body. Assuming that the condition (2.1) is satisfied for constant δ sufficiently less than unity we can determine the approximate potential $\varphi_n(x, z)$ by retaining in the expansion of the accurate expression in a series in z, a few initial terms. Of course the equation thus obtained may be used only for the values

$$z = z_0(x)+\varepsilon x, \quad \text{where} \quad 0\leqslant\varepsilon<<1.$$

To obtain the expansion required we shall use the transform $\bar{\varphi}_n(s, z)$ in the form (3.14). Retaining the first two terms of the series representing the function K_n, we shall obtain:

$$(6.1) \qquad \bar{\varphi}_n(s, z) = 2^{n-1}\frac{\bar{q}_{1n}(s)}{z^n}\frac{(n-1)!\,\mathrm{Re}'(s+i\gamma)^n}{(s^2+\gamma^2)^n}\left[1-\frac{z^2}{4(n-1)}(s^2+\gamma^2)+0(z^4)\right]$$

$$\text{for} \quad n = 3, 4, 5,...,$$

$$(6.2) \qquad \bar{\varphi}_2(s, z) = \frac{2\bar{q}_{12}(s)}{z^2}\frac{s^2-\gamma^2}{(s^2+\gamma^2)^2}\left[1-\frac{z^2}{4}(s^2+\gamma^2)+0(z^4\ln z)\right] \qquad \text{for} \quad n = 2,$$

$$(6.3) \qquad \bar{\varphi}_1(s, z) = \frac{\bar{q}_{11}(s)}{z}\frac{s}{s^2+\gamma^2}\left\{1+\frac{z^2}{4}(s^2+\gamma^2)\left[2\ln\left(\frac{z}{2}\sqrt{s^2+\gamma^2}\right)+2C-1\right]+\right.$$

$$\left.+0(z^4\ln z)\right\} \qquad \text{for} \quad n = 1$$

$$(6.4) \qquad \varphi_0(s, z) = q_{10}(s)\left\{\ln\left(\frac{z}{2}\sqrt{s^2+\gamma^2}\right)+C+\frac{z^2}{4}(s^2+\gamma^2)\left[\ln\left(\frac{z}{2}\sqrt{s^2+\gamma^2}\right)+C-1\right]+\right.$$

$$\left.+0(z^4\ln z)\right\} \qquad \text{for} \quad n = 0,$$

where $C = 0.5772\ ...$ — is Euler's constant.

Performing the inverse transformation of (6.1) to (6.4) and introducing the notations

$$(6.5) \quad Q_n(x) = \int_0^x [2(x-\xi)]^{n-1}\cos[\gamma(x-\xi)]q_{1n}(\xi)d\xi = \mathcal{L}^{-1}(n-1)!\,2^{n-1}\frac{\mathrm{Re}'(s+i\gamma)^n}{(s^2+\gamma^2)^n}\bar{q}_{1n}(s)$$

$$\text{for} \quad 1, 2, 3, \ldots$$

$$Q_0(x) = q_{10}(x),$$

we obtain the equations for the sought-for potential:

(6.6) $$\varphi_n(x, z) = \frac{1}{z^n}\left\{\left[1-\frac{z^2}{4(n-1)}\left(\gamma^2+\frac{d^2}{dx^2}\right)\right]Q_n(x)+0(z^4)\right\} \quad \text{for } n = 3, 4, 5, \ldots,$$

(6.7) $$\varphi_2(x, z) = \frac{1}{z^2}\left\{\left[1-\frac{z^2}{4}\left(\gamma^2+\frac{d^2}{dx^2}\right)\right]Q_2(x)+0(z^4 \ln z)\right\} \quad \text{for } n = 2,$$

(6.8) $$\varphi_1(x, z) = \frac{1}{z}\left\{\left[1+\frac{z^2}{2}\left(\ln\frac{z}{2}-\frac{1}{2}\right)\left(\gamma^2+\frac{d^2}{dx^2}\right)\right]Q_1(x)+\frac{z^2}{2}\left(\gamma^2+\right.\right.$$
$$\left.\left.+\frac{d^2}{dx^2}\right)\int_0^x Q_1'(\xi)G(x-\xi)d\xi+0(z^4 \ln z)\right\} \quad \text{for } n = 1,$$

(6.9) $$\varphi_0(x, z) = \left[\ln\frac{z}{2}+\frac{z^2}{4}\left(\ln\frac{z}{2}-1\right)\left(\gamma^2+\frac{d^2}{dx^2}\right)\right]Q_0(x)+$$
$$+\left[1+\frac{z^2}{4}\left(\gamma^2+\frac{d^2}{dx^2}\right)\right]\int_0^x Q_0'(\xi)G(x-\xi)d\xi+0(z^4 \ln z) \quad \text{for } n = 0,$$

where

$$G(x) = \int_0^{\gamma x}\frac{1-\cos\xi}{\xi}d\xi-\ln x.$$

For the derivation of (6.8) and (6.9), it was assumed that

(6.10) $$Q_1'(0) = Q_0(0) = Q_0'(0) = 0.$$

The potenial of the flow will be obtained in the first approximation, [3], by taking the first terms of (6.6) to (6.8) and the first term and also the first integral term of (6.9) and substituting them in (2.7). Although the first approximation is successfully used for the solution of problems of vibration of bodies as a whole, it is less suited for the analysis of local self-excited vibration of a covering (of the front portion of a body of revolution, for instance) in view of a relatively low order of the error and the incompressible character of the flow. The second approximation given above involves an error of order twice as high, takes the compressibility into consideration, and enables a more accurate description of the phase angle between the displacements of the points of the surface and the pressure, which plays an essential role in the investigation of self-excited vibrations.

Let us apply the Eqs. (6.6) to (6.9) to the case of a vibrating cone of apex angle 2ϑ. The potential $\varphi_n(x, z)$ on its surface, that is for $z = z_0(x) = \mu \operatorname{tg}\vartheta x$, will be determined by means of the above equations with error of the order:

$$0[(\mu \operatorname{tg}\vartheta)^4], \quad \text{or} \quad 0[(\mu \operatorname{tg}\vartheta)^4 \ln(\mu \operatorname{tg}\vartheta)].$$

The function $\Delta p_n(x,\mu)$, determined by (2.12) and characterizing the pressure difference in the flow takes now, on the conical surface, the form:

$$(6.11)\quad \Delta p_n[x, z_0(x)] = \frac{1}{x^n}\left\{1-\frac{\mu^2 \mathrm{tg}^2\vartheta}{4(n-1)}x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}\left(\frac{d}{dx}+i\omega\right)Q_n^*(x)$$

for $n = 2, 3, 4, \ldots,$

$$(6.12)\quad \Delta p_n[x, z_0(x)] = \frac{1}{x}\left\{1+\frac{1}{2}\mu^2 \mathrm{tg}^2\vartheta x^2\left[\ln\left(\frac{1}{2}\mu\,\mathrm{tg}\,\vartheta x\right)-\frac{1}{2}\right]\left[\left(\frac{d}{dx}+i\beta\right)^2+\right.\right.$$
$$\left.\left.+\gamma^2\right]\right\}\left(\frac{d}{dx}+i\omega\right)Q_1^*(x)+\frac{1}{2}\mu^2\,\mathrm{tg}^2\,\vartheta x\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\left(\frac{d}{dx}+i\omega\right)\int_0^x Q_1^{*\prime}(\xi)G^*(x-\xi)d\xi$$

for $n = 1,$

$$(6.13)\quad \Delta p_0[x, z_1(x)] = \left\{\ln\left(\frac{1}{2}\mu\,\mathrm{tg}\,\vartheta x\right)+\frac{1}{4}\mu^2\,\mathrm{tg}^2\,\vartheta x^2\left[\ln\left(\frac{1}{2}\mu\,\mathrm{tg}\,\vartheta x\right)-1\right]\times\right.$$
$$\left.\times\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}\left(\frac{d}{dx}+i\omega\right)Q_0^*(x)+\left\{1+\frac{1}{4}\mu^2\,\mathrm{tg}^2\,\vartheta x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}\times$$
$$\times\left(\frac{d}{dx}+i\omega\right)\int_0^x Q_0^{*\prime}(\xi)G^*(x-\xi)d\xi \qquad \text{for } n=0,$$

where

$$Q_n^*(x) = \frac{e^{-i\beta x}}{(\mu\,\mathrm{tg}\,\vartheta)^n}Q_n(x), \qquad G^*(x) = e^{-i\beta x}G(x).$$

From the boundary condition (2.9) we obtain, by rejecting the terms of orders higher than those appearing in the initial equations (6.6) to (6.9), the following equations.

A. If, in addition to the conditions (2.1), we have: $\mathrm{tg}\,\vartheta \geqslant 1/\mu^2$

$$(6.14)\quad \left\{-n+\frac{n-2}{4(n-1)}\mu^2 \mathrm{tg}^2\vartheta x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}Q_n^*(x) = \mathrm{tg}\,\vartheta\, x^{n-1}\left[\frac{dw_n(x)}{dx}+i\omega w_n(x)\right]$$

for $n = 2, 3, 4, \ldots,$

$$(6.15)\quad \left\{-1+\frac{1}{2}\mu^2\,\mathrm{tg}^2\,\vartheta x^2\left[\ln\left(\frac{1}{2}\mu\,\mathrm{tg}\,\vartheta x\right)+\frac{1}{2}\right]\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}Q_1^*(x)+$$
$$+\frac{1}{2}\mu^2\,\mathrm{tg}^2\,\vartheta x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\int_0^x Q_1^{*\prime}(\xi)G^*(x-\xi)d\xi = \mathrm{tg}\,\vartheta x^2\left[\frac{dw_1(x)}{dx}+i\omega w_1(x)\right]$$

for $n = 1,$

$$(6.16)\quad \left\{1+\frac{1}{2}\mu^2\,\mathrm{tg}^2\,\vartheta x^2\left[\ln\left(\frac{1}{2}\mu\,\mathrm{tg}\,\vartheta x\right)-\frac{1}{2}\right]\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}Q_0^*(x)+$$

$$+\frac{1}{2}\mu^2 \operatorname{tg}^2 \vartheta x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\int_0^x Q_0^{*\prime}(\xi)G^*(x-\xi)d\xi = \operatorname{tg}\vartheta x\left[\frac{dw_0(x)}{dx}+i\omega w_0(x)\right]$$

for $n = 0$.

B. If the conditions (2.1) are the only conditions satisfied:

(6.17)
$$\left\{-n-\operatorname{tg}^2\vartheta x\frac{d}{dx}+\frac{n-2}{4(n-1)}\mu^2\operatorname{tg}^2\vartheta x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}Q_n^*(x) =$$
$$= \operatorname{tg}\vartheta\left(1+\frac{1}{2}\operatorname{tg}^2\vartheta\right)x^{n+1}\left[\frac{dw_n(x)}{dx}+i\omega w_n(x)\right] \quad \text{for } n = 2, 3, 4, \ldots,$$

(6.18)
$$\left\{-1-\operatorname{tg}^2\vartheta x\frac{d}{dx}+\frac{1}{2}\mu^2\operatorname{tg}^2\vartheta x^2\left[\ln\left(\frac{1}{2}\mu\operatorname{tg}\vartheta x\right)+\frac{1}{2}\right]\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}Q_1^*(x)+$$
$$+\frac{1}{2}\mu^2\operatorname{tg}^2\vartheta x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\int_0^x Q_1^{*\prime}(\xi)G^*(x-\xi)d\xi =$$
$$= \operatorname{tg}\vartheta\left(1+\frac{1}{2}\operatorname{tg}^2\vartheta\right)x^2\left[\frac{dw_1(x)}{dx}+i\omega w_1(x)\right] \quad \text{for } n = 1,$$

(6.19)
$$\left\{1-\operatorname{tg}^2\vartheta x\ln\left(\frac{1}{2}\mu\operatorname{tg}\vartheta x\right)\frac{d}{dx}+\frac{1}{2}\mu^2\operatorname{tg}^2\vartheta x^2\left[\ln\left(\frac{1}{2}\mu\operatorname{tg}\vartheta x\right)-\frac{1}{2}\right]\left[\left(\frac{d}{dx}+\right.\right.\right.$$
$$\left.\left.\left.+i\beta\right)^2+\gamma^2\right]\right\}Q_0^*(x)-\left\{\operatorname{tg}^2\vartheta x\frac{d}{dx}-\frac{1}{2}\mu^2\operatorname{tg}^2\vartheta x^2\left[\left(\frac{d}{dx}+i\beta\right)^2+\gamma^2\right]\right\}\int_0^x Q_0^{*\prime}(\xi)G^*(x-$$
$$-\xi)d\xi = \operatorname{tg}\vartheta\left(1+\frac{1}{2}\operatorname{tg}^2\vartheta\right)x\left[\frac{dw_0(x)}{dx}+i\omega w_0(x)\right] \quad \text{for } n = 0.$$

The conditions to be satisfied by the solutions $Q_n^*(x)$ of the Eqs. (6.14) to (6.19) follow from the consideration of the equation obtained by substituting the accurate expression of the potential (3.12) in the boundary condition (2.9) as applied to the cone.

Thus it is seen that if

$$w_n(0) = 0 \text{ and } w_n'(0) \neq 0$$

we have $q_{1n}(0) \neq 0$ for every n and in order that $q_{1n}'(0) = 0$, it is necessary that

$$w_n(0) = w_n'(0) = 0.$$

Bearing in mind (6.5) and (6.10) we can therefore write

(6.20)
$$Q_n^*(0) = Q_n^{*\prime}(0) = Q_n^{*\prime\prime}(0) = \ldots = Q_n^{*(n)}(0) = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

assuming that $w_n(0) = 0$ and $w_n'(0) \neq 0$

and also
$$Q_0^*(0) = Q_0^{*\prime}(0) = 0,$$

where we must have

$$w_0(0) = w_0'(0) = 0.$$

In addition, in view of the uniqueness of the boundary condition with the accurate expression (3.12), we must have

$$Q_n^*(x) \equiv 0, \quad \text{if} \quad w_n(x) \equiv 0. \tag{6.21}$$

The condition (6.21) is in some cases necessary for the uniqueness of the solutions $Q_n^*(x)$, because, in view of the supersonic character of the flow, we have only the conditions for $x = 0$, and this point is the singular point of the equations under consideration.

We proceed now to solve the Eqs. (6.14) and (6.17). The cases $n = 1$ and $n = 0$ will be considered separately.

For $n = 2$ we obtain from (6.14):

$$Q_2^*(x) = -\frac{1}{2}\operatorname{tg}\vartheta x^3\left[\frac{dw_2(x)}{dx} + i\omega w_2(x)\right]. \tag{6.22}$$

From (6.17), after satisfying (6.20), we obtain:

$$Q_2^*(x) = -\frac{2+\operatorname{tg}^2\vartheta}{2\operatorname{tg}\vartheta}\int_0^x \frac{\xi^{k-2}}{x^k}\left[\frac{dw_2(\xi)}{d\xi} + i\omega w_2(\xi)\right]d\xi, \tag{6.23}$$

where $k = 2/\operatorname{tg}^2\vartheta$.

The Eq. (6.14) for $n = 3, 4, 5, \ldots$ may be solved by means of Bessel functions. The solution satisfying (6.20) and (6.21) has the form:

$$Q_n^*(x) = \frac{2\pi(n-1)}{(n-2)\mu^2\operatorname{tg}\vartheta}\sqrt{x}\, e^{\;i\beta x}\left\{Y_\lambda(\gamma x)\int_0^x\left[\frac{dw_n(\xi)}{d\xi} + i\omega w_n(\xi)\right]\times\right.$$
$$\left.\times\,\xi^{n\;\frac{1}{2}} e^{i\beta\xi}J_\lambda(\gamma\xi)d\xi - J_\lambda(\gamma x)\int_0^x\left[\frac{dw_n(\xi)}{d\xi} + i\omega w_n(\xi)\right]\xi^{n-\frac{1}{2}} e^{i\beta\xi}Y_\lambda(\gamma\xi)d\xi\right\}, \tag{6 24}$$

where J_λ and Y_λ are Bessel functions of the first and second kind, and

$$\lambda = \sqrt{\frac{4\,n(n-1)}{(n-2)\mu^2\operatorname{tg}^2\vartheta} + \frac{1}{4}}\,.$$

For the values of λ, for which the second integral in (6.24) becomes divergent, that integral should be replaced with the indeterminate one.

The solution of the Eq. (6.17) for $n = 3, 4, 5, \ldots$ will be obtained by means of the confluent hypergeometric function ${}_1F_1(a, b, x)$. Satisfying (6.20) and (6.21), it takes the form:

$$Q_n^*(x) = \frac{2(n-1)(2+\operatorname{tg}^2\vartheta)}{(n-2)\mu^2(b-1)\operatorname{tg}\vartheta}e^{-i(\beta-\gamma)x}\left\{x^{l-\frac{b}{2}}{}_1F_1(a, b, -2i\gamma x)\int_0^x \xi^{n-l-\frac{b}{2}} e^{i(\beta+\gamma)\xi}\times\right. \tag{6.25}$$

$$\times {}_1F_1(a',b',-2i\gamma\xi)\left[\frac{dw_n(\xi)}{d\xi}+i\omega w_n(\xi)\right]d\xi - x^{l+1-\frac{b}{2}}\,{}_1F_1(a',b',-2\,i\gamma x)\times$$

$$\times\int\limits_0^x \xi^{n+\frac{b}{2}-l-1}\,e^{i(\beta+\gamma)\xi}\,{}_1F_1(a,b,-2\,i\gamma\xi)\left[\frac{dw_n(\xi)}{d\xi}+i\omega w_n(\xi)\right]d\xi\Bigg\},$$

where

$$l=\frac{2\,(n-1)}{(n-2)\mu^2},\qquad a=\frac{1}{2}+\frac{2\,(n-1)M}{(n-2)\mu^2}+$$

$$+\frac{2}{(n-2)\mu^2\,\mathrm{tg}\vartheta}\sqrt{(n-1)(n-2)\mu^2\left(n+\frac{1}{2}\,\mathrm{tg}^2\vartheta\right)+\mathrm{tg}^2\vartheta\left[(n-1)^2+\frac{\mu^4}{16}(n-2)^2\right]},$$

$$b=1+\frac{4}{(n-2)\mu^2\,\mathrm{tg}\vartheta}\sqrt{(n-1)(n-2)\mu^2\left(n+\frac{1}{2}\,\mathrm{tg}^2\vartheta\right)+\mathrm{tg}^2\vartheta\left[(n-1)^2+\frac{\mu^4}{16}(n-2)^2\right]},$$

$$a'=1+a-b,\qquad b'=2-b.$$

For values of $l+b/2$ for which the first integral in (6.25) becomes divergent, that integral should be replaced with an indeterminate one. The form (6.25) of solution of the Eq. (6.17) is valid assuming that $b\neq 0,\pm 1,\pm 2,\ldots$ For $b=0,\pm 1,\pm 2,\ldots$, the solution will have a different form, which can easily be determined.

On substituting (6.22) to (6.25) in (6.11) and then in (2.11), we shall obtain the sought-for pressure difference on the surface of the vibrating cone.

References

[1] Ф. И. Франкль, Е. А. Карпович, *Газодинамика тонких тел*, Москва 1948.

[2] Е. А. Красильщикова, *Крыло конечного размаха в сжимаемом потоке*, Москва 1952,

[3] J. W. Miles. *The potential theory of unsteady supersonic flow*, Cambridge 1959.

[4] W. H. Dorrance. *Nonsteady supersonic flow about pointed bodies of revolution*, J.A.S.. 8, **18** (1951).

[5] Н. Ф. Краснов, *Аэродинамика тел вращения*, Москва 1958.

[6] Г. И. Копзон, *Вибрации тонкостенных упругих тел в потоке газа*, Докл. АН СССР 2, **107**(1956).

[7] M. Holt, S. L. Strack, *Supersonic panel flutter of a cylindrical shell of finite length*, IAS Paper No 60-22, 1960.

[8] G. N. Watson. *A Treatise on the theory of Bessel functions*, Cambridge 1944.

Streszczenie

ZLINEARYZOWANY OPŁYW NADDŹWIEKOWY DRGAJĄCEJ POWIERZCHNI CIAŁ OBROTOWYCH

W pracy wyznaczono postać potencjału naddźwiękowego opływu zewnętrznego drgającej harmonicznie powierzchni ciał obrotowych. W przypadku ogólnym problem opływu sprowadzono do równania całkowego Volterry drugiego rodzaju.

Dla cylindra drgającego w opływie podano zależność ciśnienia od składowej normalnej przemieszczenia w postaci asymptotycznego rozwinięcia względem odwrotności liczby Macha. Wyprowadzono drugie liniowe przybliżenie potencjału opływu ciał smukłych zaostrzonych i zastosowano dla wyznaczenia ciśnienia na drgającym stożku.

Резюме

ЛИНЕАРИЗОВАННОЕ СВЕРХЗВУКОВОЕ ОБТЕКАНИЕ ПОВЕРХНОСТИ ТЕЛ ВРАЩЕНИЯ

Определяется форма потенциала сверхзвукового наружного обтекания колеблющейся гармонически поверхности тела вращения. В общем случае задача сводится к интегральному уравнению Вольтерри второго рода. Для цилиндра колеблющегося при обтекании дается зависимость давления от нормальной составляющей перемещения в виде асимптотического разложения по отношению к обратности числа Маха. Выводится второе линейное приближение потенциала обтекания тонких заостренных тел, которое применяется для определения давления на колеблющимся конусе.

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A TWO-TENSOR METHOD FOR INVESTIGATING NONLINEAR SYSTEMS

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1. Introduction

Many technical problems are solved on the basis of the theory of ordinary differential equations. Such problems are, among others, those of linear elastic vibration, electrical vibration, stability of equilibrium of motion, orbital stability, engineering stability, and stability of transitory processes of automatic control. The solution of each of these problems requires a separate method. If a system of differential equations is linear with constant coefficients, the solution presents no essential difficulty. In the case of a system of nonlinear equations, or linear equations with variable coefficients, approximate or qualitative methods are most often used, based on topological properties of the solution of systems of ordinary differential equations.

The behaviour of the solution of a system of two equations on the phase plane was studied by H. Poincaré, [1], by means of contactless curves. These are curves having no contact points with the integral curves determined by the system of equations. If a contactless curve is closed all its points are points of entry or points of issue. If such curves can be determined for a nonlinear system it is easy to describe the behaviour of the solution of this system. Also A. Lapunov, [2], uses surfaces of which the points are either entry or issue points.

The problem of the behaviour of the solutions in the case where there are entry or issue points on closed surfaces was analysed by T. Ważewski in 1945, [3]. Using the notion of retract, Ważewski arrived at a number of conclusions in the case where all the issue points on a closed surface are rigorous issue points.

In practice, the determination of contactless curves or Lapunov surfaces whose points are only those of rigorous issue is, in most common cases, very complicated. In the present paper, a method is proposed for investigating the behaviour of the solutions of autonomic systems by means of spherical surfaces with their centres at the singular point. On the surface of such a sphere may appear entry, issue and slip points. The behaviour of the solution may be deduced from the properties of two tensors, doubly covariant and contravariant of valence (0,2). The method is illustrated by a few examples.

2. Basic Assumptions and Dynamic Interpretation of the Method

Let us consider an autonomic system

$$\dot{X} = F(X), \tag{2.1}$$

where X is a vector with coordinates $(X_1,\dots,X_n)$ and F — a vector with coordinates $(F_1,\dots,F_n)$. The assumptions are as follows.

Assumption 1. The functions F_i are of the C^1 class in an n-dimensional Euclidean space E_n.

Assumption 2. The point $(0,\dots,0)$ is an isolated singular point.

Assumption 3. The Jacobian of the functions F_i in the space E_n is different from zero:

$$I = \left\| \frac{\partial F_i}{\partial X_j} \right\| \neq 0.$$

Assumption 4. The transformation $y = F(X)$ has its inverse in E_n. The inverse transformation will be denoted by $X = \varphi(y)$.

Let us denote the partial derivatives $\partial F_i/\partial X_i$, with the substitution for X_j of the inverse transform $X_j = \varphi_j(y_1,\dots,y_n)$ by $a_{ij}(y_1,\dots,y_n)$, and consider the system of equations

$$\dot{y} = a(y)y, \tag{2.2}$$

where $a(y)$ is a $n \times n$—matrix of elements $a_{ij}(y_1,\dots,y_n)$. With these assumptions it will be shown in Sec. 3 that the study of the behaviour of the solutions of the system (2.1) reduces to that of the solution of the system (2.2).

The form of the solutions of (2.2) in the phase space E_n depends on the properties of two tensors: the doubly covariant tensor obtained from the scalar product of the distance vector $\boldsymbol{\rho}$ with the coordinates $(y_1,\dots,y_n)$ and the velocity vector $\mathbf{V}$ with the coordinates $(\dot{y}_1,\dots,\dot{y}_n)$ and a doubly contravariant tensor constituting a bivector of the vectors $\mathbf{V}$ and $\boldsymbol{\rho}$.

The method for testing the solution of (2.2) will be interpreted for a system of three equations in the phase space E_3. Let us consider the system of equations

$$\dot{X}_i = F_i(X_1, X_2, X_3) \qquad (i = 1, 2, 3). \tag{2.3}$$

The system (2.3) may be interpreted as an equation of motion in the space E_3 of a material point p with unit mass. The trajectory of the point p passing through p_0 will be denoted by $f(p_0,t)$, and the velocity vector of p by $\mathbf{V}$.

The coordinates of the vector $\mathbf{V}$ are determined by the right-hand members of (2.3). The kinetic energy of p is:

$$E = \frac{1}{2}(\dot{X}_1^2+\dot{X}_2^2+\dot{X}_3^2). \tag{2.4}$$

Let us differentiate both members of (2.3) with respect to time, substitute $\dot{X}_i = y_i$, and apply the transformation and the notations (4). We obtain a system of equations in the form (2.2), that is

$$\dot{y}_i = \sum_{j=1}^{3} a_{ij}(y_1, y_2, y_3) y_j \qquad (i=1, 2, 3); \tag{2.5}$$

the system (2.5) determines the motion of the point P with unit mass in the space E_3. The trajectory of the point P passing through P_0 will be denoted by $g(P_0, t)$, and the velocity vector by $\mathbf{V}(y_1, y_2, y_3)$

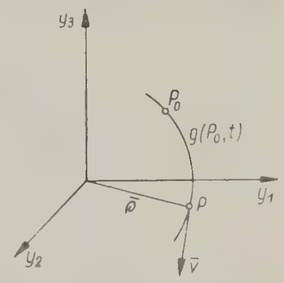

Fig. 1

The distance vector of the point P from the origin has the coordinates (y_1, y_2, y_3) and will be denoted by $\boldsymbol{\rho}$.

Let the point p_1 be mapped (Assumption 4) into P_0. The trajectory $f(p_0,t)$ is mapped then into $g(P_0,t)$, and the kinetic energy (2.4) is half the square of the distance of P from the origin. The derivative of the kinetic energy with respect to time is expressed by the scalar product of $\boldsymbol{\rho}$ and $\mathbf{V}$

$$\frac{d}{dt}(E) = \boldsymbol{\rho}\mathbf{V} = y_1\dot{y}_1 + y_2\dot{y}_2 + y_3\dot{y}_3. \tag{2.6}$$

Substituting for $\dot{y}_i$ the right-hand members of the equations (2.5) we obtain a function of the variables y_1, y_2, y_3 denoted by Φ:

$$\Phi(y_1, y_2, y_3) \stackrel{\text{def.}}{=} \boldsymbol{\rho}\mathbf{V} = \sum_{i,j=1}^{3} b_{ij} y_i y_j, \tag{2.7}$$

where the coefficients $b_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ are functions of y_1, y_2, y_3.

If the function (2.7) has a constant sign along the trajectory $g(P_0, t)$, the kinetic energy of p is a monotonic function along the trajectory $f(p_0,t)$. In Sec. 3 it will be shown that in this case the point p tends to the singular point or its distance from the origin tends to infinity.

Note. In the following the expression (a the trajectory tends to the singular point or to infinity) will mean that the distance of the point p on the trajectory from the origin tends to zero or to infinity.

The sign of the function Φ is decided upon by the matrix $\|b_{ij}\|$ which is the symmetrized Jacobian matrix of the functions F_i with the inverse transform according to the Assumption 4. If the matrix $\|b_{ij}\|$ is of a definite sign along the trajectory $g(P_0,t)$ the function Φ has a constant sign. It may happen that the matrix $\|b_{ij}\|$ is of a definite sign in the entire space E_3. In this case Φ has a constant sing along every trajectory $g(P_0,t)$ and the form of these trajectories can

easily be determined. Every trajectory either approaches the singular point or is of increasing distance from it, thus tending to infinity.

Such in this case is the geometrical interpretation of the form of the trajectory.

Let us consider a sphere with the centre at the singular point and with any radius ϱ:

$$\varrho^2 = y_1^2+y_2^2+y_3^2. \tag{2.8}$$

The velocity vector **V**, tangent to the trajectory $g(P_0,t)$, is determined at every point M of the spherical surface.

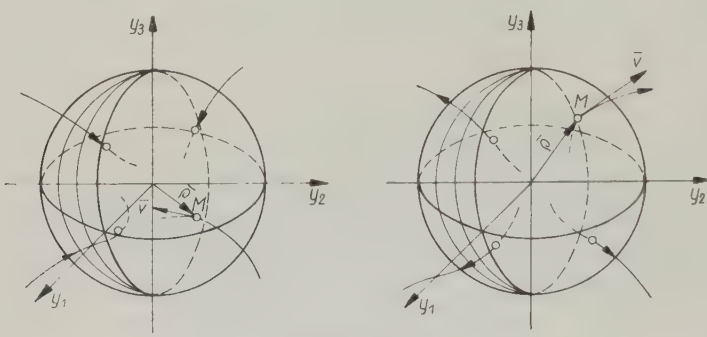

Fig. 2

The scalar product of the vector normal to the surface of the sphere at the point M and the vector **V** is the function Φ. If at every point of the sphere (2.8) the function Φ is negative, the point P moving on the trajectory $g(P_0, t)$ enters into the spherical region. Every point on the spherical surface (2.8) is an entrance point. If this property holds good for every ϱ that is if the function Φ is negative in E_3, every trajectory tends to the singular point.

Similarly, if Φ is positive, every point on the spherical surfaces (2.8) is an exit point, and the trajectories tend to infinity (Fig. 2).

The case where the function Φ may change sign in the space E_3 is more involved. In this case, entrance and exit points may appear on a spherical surface and the kinetic energy (2.4) must be non-monotonic on certain trajectories.

Here, three regions may be discerned in the space E_3: the region ω^+ where the function Φ is positive, the region ω^- where it is negative and the boundary region ω^0. The points on a spherical surface (2.8) belonging to the region ω^0 are points of internal or external slip. The point P_0 moving along the trajectory $g(P_0,t)$ may remain, starting from a certain time t, in the region ω^+ or ω^-. Then, its distance from the origin increases or decreases. The point P may pass from ω^+ to ω^-, and vice-versa, a finite or an infinite number of times. In these cases the form of the trajectory is tested by means of the moment of momentum of P about the origin. The vector of the moment of momentum is a vector product of the velocity vector

$\mathbf{V}$ and the distance vector $\boldsymbol{\rho}$. The coordinates of the moment of momentum are expressed thus:

$$\psi_{ij} = \dot{y}_j y_i - y_j \dot{y}_i \qquad (i, j = 1, 2, 3). \tag{2.9}$$

The functions ψ_{ij}, after replacing $\dot{y}_i$ with the right-hand members of (2.5), are functions of the coordinates (y_1, y_2, y_3).

Let us assume that all the functions ψ_{ij} become zero in a certain region Δ. From the definition of the functions ψ_{ij} it follows that in the region Δ the vector $\mathbf{V}$ has the direction of the vector $\boldsymbol{\rho}$ and an attraction or repulsion zone will appear in the region Δ, depending on whether the vectors $\mathbf{V}$ and $\boldsymbol{\rho}$ have the same or opposite sense. Let us assume that the attraction zone is contained in the regions ω^-, where the function Φ is negative and the repulsion zone — in the region ω^-, where Φ is positive. If the regions Δ, ω^+ and ω^- are detarmined, we can, as will be shown in the examples, determine the form of the trajectory $g(P_0, t)$. It may happen that the trajectory $g(P_0, t)$ lies in the region Δ. Along such a trajectory the following equation is valid:

$$\mathbf{V} = \lambda \boldsymbol{\rho}, \tag{2.10}$$

where the function λ depends on the location of the point P on the trajectory. From the assumptions (2) and (4), it follows that the vectors $\mathbf{V}$ and $\boldsymbol{\rho}$ may become zero only at the singular point and, therefore, the function $\lambda(y_1, y_2, y_3)$ is continuous and has a constant negative or positive sign.

From (2.10) it follows that the trajectory $g(P_0, t)$ is, in this case, determined by the equations:

$$y_i = y_i^0 \, e^{\int_{t_0}^{t} \lambda(y_1, y_2, y_3) dt} \qquad (i = 1, 2, 3). \tag{2.11}$$

The trajectory (2.11) may be entirely inside the region ω^+ or ω^-. This depends on the sign of $\lambda(y_1, y_2, y_3)$.

Let us consider the other case, where the vectors of the moment of momentum (2.9) are different from zero in the space E_3. The vectors $\mathbf{V}$ and $\boldsymbol{\rho}$ are in this case different from zero and determine the plane where the point P remains moving along the trajectory $g(P_0, t)$. It will be shown in Sec. 3 that as between the functions ϱ, Φ, ψ the following relation is valid:

$$\varrho = \varrho_0 \, e^{\int_0^{\varphi} \frac{\Phi}{\psi} d\varphi}, \tag{2.12}$$

where $\boldsymbol{\rho}_0$ is the modulus of the vector $\boldsymbol{\rho}$ at the initial time $t = t_0$, ψ — the modulus of the vector of the moment of momentum, φ-the rotation angle of the vector $\boldsymbol{\rho}$ in the plane determined by $\mathbf{V}$ and $\boldsymbol{\rho}$. The investigation of the form of the trajectory reduces in this case to that of the sign of the integral in the Eq. (2.12). The functions under the integration sign depend on ϱ and φ. In spite of this relation,

the estimate of the ratio of Φ to ψ is often possible. In practice, it is more convenient to examine the form of the trajectory $g(P_0, t)$, in the case under consideration, by projecting the point P moving along the trajectory $g(P_0, t)$ on the planes (y_1, y_2), (y_1, y_3), (y_3, y_1), the form of the trajectory of P being deduced from that of the projections.

3. Mathematical Justification of the Method

Let the autonomic system be described by the system of ordinary differential equations

$$\dot{X} = F(X), \tag{3.1}$$

where $X = (X_1,...,X_n)$, $F = (F_1,...,F_n)$, as in Sec. 1. Let us make the Assumptions 1, 2, 3, 4 of the preceding section and consider the system of equations

$$\dot{y} = a(y)\, y, \tag{3.2}$$

where $a(y)$ is an $n\times n$—matrix. Let us denote, as before, the trajectory passing through the point $p_0 \in E_n$ at the time $t = t_0$ by $f(p_0,t)$, and that passing through $P_0 \in E_n$ by $g(P_0,t)$. The point p_0 is mapped, by the transformation defined in the Assumption 4, into P_0. On the basis of this assumption, we can write:

$$g(P_0, t) = F\,[f(p_0, t)] \quad \text{and} \quad f(p_0, t) = \varphi[g(P_0, t)]. \tag{3.3}$$

We proceed now to prove the following theorems:

Theorem I. If the trajectory $f(p_0, t)$ is bounded, the trajectory $g(P_0, t)$ is also bounded.

Theorem II. If the trajectory $f(p_0, t)$ tends to the singular point, the trajectory $g(P_0, t)$ also tends to the singular point.

Theorem III. If the trajectory $f(p_0, t)$ tends to infinity, the trajectory $g(P_0, t)$ also tends to infinity.

P r o o f of Theorem I. Let us assume that the trajectory $f(p_0, t)$ is bounded—that s the distance of p on $f(p_0, t)$, from the origin is bounded.

It follows that there exists an $r>0$ satisfying the inequality

$$\varrho[0,f(p_0, t)] \leqslant r \qquad \text{for} \quad t \in [t_0, \infty], \tag{3.4}$$

where $\varrho[0,f(p_0, t)] = \sqrt{\sum_{i=1}^{n} x^2}$ and $(x_1,\ldots,x_n)$ are the coordinate of p.

By virtue of the Assumption 1, the functions $F_i(X_1,\ldots,X_n)$ are continuous in E_n and bounded in the closed region (3.4) Therefore, there exists an $R>0$ such that

$$F[f(p_0, t)] \leqslant R, \tag{3.5}$$

if $f(p_0, t)$ lies in the region (3.4). From (3.5) and (3.3) it follows that $g(P_0, t)$ is bounded.

Proof of Theorem II. Let us assume that the trajectory $f(p_0, t)$ tends to the singular point—that is the distance of the point p on the trajectory $f(p_0, t)$ tends to zero. From the definition of the distance of the point p on the trajectory it follows that all the coordinates X_i tend to zero. From the continuity of the function $F_i(X_1,...,X_n)$ and from the Assumption 4, it follows that $F(X)$ tends to $F(0)$ and $F(0) = 0$. By virtue of (3.3), the trajectory $g(P_0, t)$ tends to the singular point. The inverse theorems of I and II are also valid and the proofs are similar.

Indirect Proof of the Theorem III. Let us assume that the trajectory $f(p_0, t)$ tends to infinity and that the trajectory $g(P_0, t)$ is bounded. Bearing in mind the inverse theorem of I, we reach a contradiction, because, by virtue of (3.3), the trajectory $g(P_0, t)$ becomes the trajectory $f(p_0, t)$.

With the theorems just proved, the test of the solutions of the system of equations (3.1) for boundedness and for the quality of tending to zero, reduces to the investigation of the form of the solutions of the system of equations (3.2).

3.1. The doubly Covariant tensor and the sufficient condition for the straightness of the system (3.2). Let us construct a symmetric Jacobian matrix by adding to the matrix $a(y)$ the transposed matrix $a^T(y)$ and dividing by 2:

$$b(y) \overset{\text{def}}{=} \frac{1}{2}[a(y)+a^T(y)] = \|b_{ij}\|. \tag{3.6}$$

The following additional assumption will be made for the matrix $b(y)$

$$\|b_{ij}\| \neq 0 \qquad \text{in the space } E_n. \tag{3.7}$$

Let us consider a point P on the trajectory $g(P_0, t)$. The vector of the distance $\boldsymbol{\rho}$ of the point P from the singular point has the coordinates $(y_1,\dots, y_n)$, and the vector of the velocity $\mathbf{V}$ has the coordinates $(\dot{y}_1,\dots, \dot{y}_n)$. The scalar product of $\boldsymbol{\rho}$ and $\mathbf{V}$ is

$$\Phi = \sum_{i=1}^{n} y_i \dot{y}_i = \boldsymbol{\rho}\mathbf{V}. \tag{3.8}$$

Replacing $\dot{y}_i$ with the right-hand members of (3.2), the function (3.8) is expressed by

$$\Phi(y_1,\dots, y_n) = \sum_{i,j=1}^{n} b_{ij} y_i y_j. \tag{3.9}$$

Let us consider an arbitrary point P_0 in the space E_n, and calculate the coefficients b_{ij} at the point P_0:

$$b_{ij}(y_1^0,\dots, y_n^0) = b_{ij}^0. \tag{3.10}$$

Substituting b_{ij}^0 in (2.12), we obtain a homogeneous quadratic form

$$\Phi^0(y_1,\dots, y_n) = \sum_{i,j=1}^{n} b_{ij}^0 y_i y_j. \tag{3.11}$$

It is known that the coefficients of a homogeneous quadratic form are transformed as the coordinates of a doubly covariant tensor. It follows that the geometric entity (3.11) is a doubly convariant tensor. At every point of the space E_n, we can construct a geometric entity (3.11), thus determining a tensor field in E_n. Let us consider the case where the matrix $b(y)$ is determined at every point of the space E_n. Let us suppose that the matrix $b(y)$ is positively definite at every point of the space E_n. The right-hand member of (3.9) can, in this case, be reduced to a sum of squares of which the coefficients are positive. The function $\Phi(y_1,..., y_n)$ has the plus sign in the entire space E_n except the singular point where it is zero. It will be shown that in this case the system (3.1) is straightened. To this end the Barbaskin theorem, [16], will be used. If in an open region contained in E_n there exists a function $u(x_1,..., x_n)$ having constant partial derivatives, and if there exists a number $k^2>0$ such that the inequality

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} F_i \geqslant k^2, \tag{3.12}$$

holds, the system (3.1) is straightened. In our case, the function $u(x_1,..., x_n)$ will be determined by the equation:

$$u(x_1,..., x_n) = \frac{1}{2} \sum_{i=1}^{n} [F_i(x_1,..., x_n)]^2 . \tag{3.13}$$

It can easily be verified that

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} F_i = \Phi(y_1,..., y_n), \tag{3.14}$$

on substituting $y_i = F_i$. Let us take any constants $\varrho_2>\varrho_1>0$, and determine the region $\overline{G}_y$ by means of the inequality

$$0<\varrho_1^2 \leqslant \sum_{i=1}^{n} y_i^2 < \varrho_2^2. \tag{3.15}$$

In the region $\overline{G}_y$, the function Φ is continuous, positive and has a positive minimum which will be denoted by $k^2>0$:

$$\Phi(y_1,..., y_n) \geqslant k^2>0. \tag{3.16}$$

The inequality (3.13) holds also in the open region G_y. The assumption 4 makes the region G_y into the region G_x open and contained in E_n. On the basis of (3.16) and (3.14) the following inequality holds in the open region G_x:

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} F_i \geqslant k^2>0. \tag{3.17}$$

Hence, on the basis of the theorem just cited, the system (3.1) is straightened. It will be shown, in addition, that the trajectories $g(P_0, t)$ tend to infinity. The square of the distance of any point P on the trajectory from the origin is:

$$\varrho^2 = \sum_{i=1}^{n} y_i^2. \tag{3.18}$$

It can easily be seen that:

$$\frac{d\varrho^2}{dt} = 2\Phi(y_1, \ldots, y_n). \tag{3.19}$$

The function $\Phi(y_1, \ldots, y_n)$ is positive in E_n, therefore the function ϱ^2 is an increasing function. Let us fix $t = t_1$ and determine the value of the function $\varrho^2(t)$ at t_1 by ϱ_1^2. Let us take any $t > t_0$ and apply the theorem on the mean value of the function $\varrho^2(t)$ of one variable:

$$\frac{\varrho^2(t) - \varrho_1^2}{t - t_1} = \left[\frac{d\varrho^2}{dt}\right]_{t=t^*}, \qquad \varrho^2(t) > \varrho_1^2 > 0, \tag{3.20}$$

where $t^* \in [t_1, t]$.

If t tends to infinity, the denominator of the fraction (3.20) tends to infinity. It will be shown that the numerator tends also to infinity—that is $\varrho^2(t)$ increases indefinitely. Suppose that $\varrho^2(t)$ is bounded. The fraction (3.20) tends to zero, and the derivative of $\varrho^2(t)$ tends also to zero. On the basis of (3.19), the function $\Phi[y_1, (t), \ldots, y_n(t)]$ would tend to zero, but the function Φ is continuous in E_n and becomes zero only at the singular point. Hence it is concluded that $y_i(t)$ tends to zero and, by virtue of (3.18), $\varrho^2(t)$ tends also to zero, which is impossible in view of (3.20). The other case, where the matrix $b(y)$ is negatively defined, may be located in an analogous manner.

In this case, the system (3.1) is also straightened, and the trajectories tend to the singular point. From the above reasoning, we have the

Theorem IV. If the matrix $b(y)$ of the covariant coordinates of the tensor (3.11) has, at every point of the space E_n a definite sign, the system (3.1) is straightened and the trajectories of the system tend to infinity if the sign of the matrix is positive, and to the singular point if is negative.

The geometrical interpretation of the doubly covariant tensor (3.11) is such. Let us consider an arbitrary point P_0 in E_n, and calculate at this point the coefficients b_{ij} as before. Let us substitute the coordinates of P_0 in the right-hand member of (3.11). We obtain a geometrical entity which is an ellipsoid described by the equation

$$\Phi(y_1^0, \ldots, y_n^0) = \sum_{i,j=1}^{n} b_{ij}^0 y_i y_j. \tag{3.21}$$

For linear systems, the coefficients b_{ij} are constants independent of the point P_0. The ellipsoid (3.21) has constant principal directions. It is concluded that linear

vibration is such that only the length of the axes of the ellipsoid (3.21) varies, the principal directions remaining unchanged. For nonlinear vibrations, the principal directions of the ellipsoid vary with the point P_0.

Let us consider now the case where the matrix $b(y)$ is not of definite sign at some points of the space E_n or in the entire space E_n. The form of the solutions of the system of equations (3.2) is tested by means of a doubly contravariant tensor.

3.2. The doubly contravariant tensor. Let us form of the vectors $\boldsymbol{\rho}$ and $\mathbf{V}$, the bivector $\boldsymbol{\psi}$. The coordinates of the bivector $\boldsymbol{\psi}$ are determined by the equation

$$\psi_{ij} = \dot{y}_j y_i - y_j \dot{y}_i \qquad (i, j = 1, \dots, n, \quad i \neq j), \tag{3.22}$$

or can be expressed as the minors of the matrix

$$\begin{vmatrix} \dot{y}_1 \cdots\cdots \dot{y}_n \\ y_1 \cdots\cdots y_n \end{vmatrix}. \tag{3.23}$$

The bivector of which the coordinates are determined by (3.22), is called a doubly contravariant tensor. The condition of the bivector $\boldsymbol{\psi}$ becoming zero, is necessary and sufficient for the vectors $\boldsymbol{\rho}$ and $\mathbf{V}$ to be parallel. In this case the terms in the first row of the matrix (3.2) are proportionate to those in the second row. Let us assume that the bivector $\boldsymbol{\psi}$ becomes zero on the trajectory $g(P_0, t)$. The vectors $\boldsymbol{\rho}$ and $\mathbf{V}$ are parallel on this trajectory, and from the parallel condition we obtain the relation between the coordinates of these vectors

$$\dot{y}_i = \lambda y_i, \tag{3.24}$$

where λ is a function of the location of P on the trajectory $g(P_0, t)$.

From Eq. (3.24) we obtain:

$$y_i = y_i^0 e^{\int_{t_0}^{t} \lambda dt} . \tag{3.25}$$

Let us examine the conditions of existence of a function λ satisfying (3.24). Substituting the right-hand members of (3.2) in (3.24) for $\dot{y}_i$, we obtain the system of equations:

$$\begin{cases} (a_{11} - \lambda) y_1 + a_{12} y_2 + \dots a_{1n} y_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1} y_1 + a_{n2} y_2 + \dots (a_{nn} - \lambda) y_n = 0, \end{cases} \tag{3.26}$$

where a_{ij} are functions of the variables $(y_1, \dots, y_n)$. The condition necessary for the existence of the function λ is that the determinant

$$\begin{vmatrix} a_{11} - \lambda \dots a_{1n} \\ \dots\dots\dots\dots \\ a_{n1} \dots a_{nn} - \lambda \end{vmatrix} = 0, \tag{3.27}$$

becomes zero. From the Eq. (3.27), it follows that the function λ (if it exists) must be different from zero. The free term in the Eq. (3.27) is the Jacobian (c), of which

it has been assumed that it is different from zero in E_n. It follows that if there exists a function λ, satisfying (3.27) and (3.26), the trajectory (3.25) tends to the singular point for negative λ and to infinity for positive λ.

It can easily be shown that the function Φ has, along the trajectory (3.25). a constant sign identical with that of λ. For this it suffices to substitute (3.24) in (3.8).

Let us consider now the case where the bivector $\boldsymbol{\psi}$ is different from zero in the space E_n, except at the singular point where it is zero. In this case, the bivector $\boldsymbol{\psi}$ determines, at every point P of the trajectory, a two-dimensional hyperplane in the space E_n, passing through the singular point. In this plane lie the vectors $\mathbf{V}$ and $\boldsymbol{\rho}$ Fig. 3).

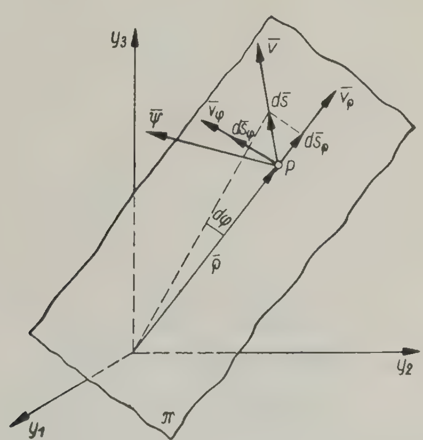

Fig. 3

The vector $\mathbf{V}$ can be resolved in this plane into two vectors: $\mathbf{V}_\varphi$ normal to $\boldsymbol{\rho}$ and $\mathbf{V}_\varrho$, whose direction coincides with that of $\boldsymbol{\rho}$.

Let us introduce on this plane an orthogonal reference frame, determined by the vectors $\boldsymbol{\rho}$ and $\mathbf{V}_\varphi$, and having its origin at. The coordinate of the vector $\mathbf{V}_\varphi$ is always positive. The coordinate of the vector $\mathbf{V}_\varrho$ is positive or negative depending on whether the vectors $\mathbf{V}_\varrho$ and $\boldsymbol{\rho}$ are of the same or opposite sense, and may be determined by the equation.

$$V_\varrho = \frac{d\varrho}{dt}, \tag{3.28}$$

The elementary displacement of the end of the vector $\boldsymbol{\rho}$ in the direction of $\mathbf{V}$ may be represented as a geometric sum of the elementary displacements ds_φ and ds_ϱ along the axes of the reference frame assumed. The coordinate of the vector ds_φ will be computed by means of the equation

$$ds_\varphi = \varrho d\varphi, \quad \text{or} \quad ds_\varphi = V_\varphi dt, \tag{3.29}$$

where $d\varphi$ is the elementary rotation angle of the vector $\boldsymbol{\rho}$ about the singular point. From (3.29) we have:

$$\varrho d\varphi = V_\varphi dt, \quad \text{or} \quad \frac{d\varphi}{dt} = \frac{V_\varphi}{\varrho}. \tag{3.30}$$

It can easily be verified that $V_\varphi \varrho = |\psi|$ and, on substituting in (2.35), we obtain:

$$\frac{d\varphi}{dt} = \frac{|\psi|}{\varrho^2}. \tag{3.31}$$

From (3.31) it follows that the angle φ is a monotonic function if the bivector ψ is different from zero outside the singular point. Let us compute the ratio of the coordinates V_ϱ and V_φ:

$$\frac{V_\varrho}{V_\varphi} = \frac{d\varrho/dt}{d\varphi/dt}. \tag{3.32}$$

Bearing in mind the relations

$$\Phi = \varrho\frac{d\varrho}{dt}, \quad |\psi| = \varrho^2\frac{d\varphi}{dt}, \tag{3.33}$$

we obtain from (3.32)

$$\frac{d\varrho/dt}{\varrho(d\varphi/dt)} = \frac{\Phi}{|\psi|}. \tag{3.34}$$

Rearranging (2.39), we obtain the differential equation

$$\frac{1}{\varrho}\frac{d\varrho}{dt} = \frac{\Phi}{|\psi|}\frac{d\varphi}{dt}, \tag{3.35}$$

of which the solution is

$$\varrho = \varrho_0 e^{\int_0^\varphi \frac{\Phi}{|\psi|} d\varphi}, \tag{3.36}$$

where ϱ_0 is the modulus of the initial vector $\boldsymbol{\rho}_0$ and the angle φ is measured from $\boldsymbol{\rho}_0$. The functions Φ and $|\psi|$ in the Eq. (3.36) depend on the coordinates of the point P on the trajectory $g(P_0, t)$. In many cases, an appraisal of the ratio of these functions is possible, and (3.36) may be used for testing the boundedness or the periodicity of the solutions of the system of equations (3.2).

The following conclusions may be drawn from the above reasoning, concerning the form of the solutions of the system (3.2). In the case where the matrix $b(y)$ is indeterminate at some points or in the entire space E_n, we have:

Conclusion I. If the bivector ψ becomes zero over a certain region, this region constitutes an attraction or repulsion zone depending on whether it is contained in the region $\Phi < 0$ or $\Phi > 0$. The trajectory contained in the region $|\psi| = 0$ has the form (2.30).

Conclusion II. If the bivector ψ becomes zero at the singular point, only, the investigation of the form of the solutions reduces to that of the integral in (3.36).

In practice, it is more convenient, in that case, to examine the form of the trajectory of the point P by projecting P on the coordinate planes. From the form of the trajectories of the projections, some conclusions may be drawn on the spatial form of the trajectory.

4. Applications

Example 1. Vibration of a dynamic damper. The equation of motion of the dynamic damper represented at Fig. 4. with nonlinear damping characteristic can be written in the form

(4.1) $$\ddot{x}+x+a\dot{x}+b\dot{x}^2+c\dot{x}^3 = 0,$$

where a, b, c are constant parameters of the ysstem and $a > 0$.

Substituting $x = x_1$, $\dot{x} = x_2$ we obtain the system of equations:

(4.2) $$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1-ax_2-bx_2^2-cx_2^3.$$

Denoting $y_1 = x_2$, $y_2 = -x_1-ax_2-bx_2^2-cx_2^3$, we obtain the system of equations:

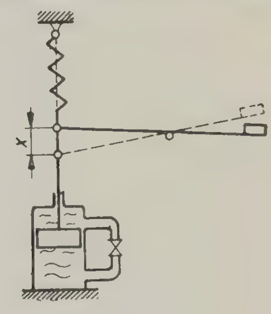

Fig. 4

(4.3) $$\dot{y}_1 = y_2, \quad \dot{y}_2 = -y_1-(a+2by_1+3cy_1^2)y_2.$$

The Jacobian of the system (4.2) is equal to unity. Let us calculate the function

(4.4) $$\Phi = -y_2^2(a+2by_1+3cy_1^2).$$

If $b^2 < 3ac$, the function Φ is non-positive in the entire plane (y_1, y_2). By virtue of the Theorem I, every trajectory of the system (4.3) tends to the singular point. Let us investigate the coordinates of the tensor ψ. Two coordinates of the tensor are zero, the third is expressed by:

(4.5) $$\psi = -y_1^2-y_1y_2(a+2by_1+3cy_1^2)-y_2^2.$$

If

(4.6) $$a+2by_1+3cy_1^2 \leqslant 4,$$

the function ψ is negatiwe.

The inequality (4.6) holds, if $b^2 > 3c(a-2)$ and y_1 satisfies the inequality:

(4.7) $$-\frac{b+\sqrt{b^2-3c(a-2)}}{3c} \leqslant y_1 \leqslant -\frac{b-\sqrt{b^2-3c(a-2)}}{3c}.$$

Outside the interval (4.7), the function ψ becomes zero on the curves:

(4.8) $$y_1+\frac{y_2}{2}(a+2by_1+3cy_1^2) \mp \frac{y_2}{2}\sqrt{(a+2by_1+3cy_1^2)^2-4} = 0.$$

A field of linear elements is distributed along the curves (4.8) thus constituting an attraction zone.

The form of the solutions is illustrated in Fig. 5.

Let us consider now the case of $b^2 - \geqslant 3ac$. In this case the form of the solutions for $c > 0$ and $c < 0$ is different. For $c > 0$ the function Φ is positive, if y_1 satisfies the inequality:

$$-\frac{b+\sqrt{b^2-3ac}}{3c} < y_1 < -\frac{b-\sqrt{b^2-3ac}}{3c}. \tag{4.9}$$

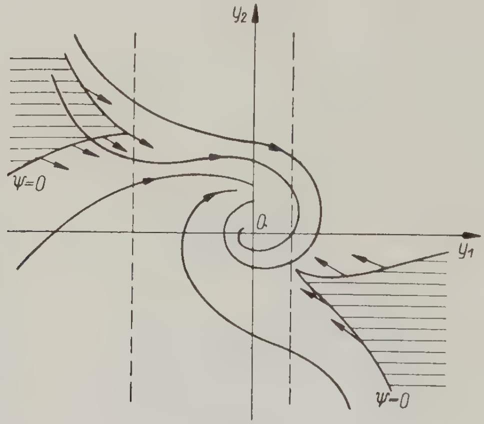

Fig. 5

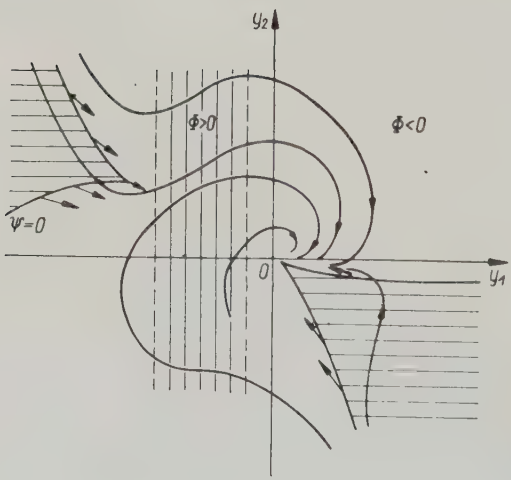

Fig. 6

The function ψ becomes zero on the curves (4.8), an attraction zone being formed as before. The form of the solutions is shown at Fig. 6.

For $c < 0$ the function Φ is positive outside the interval (4.9). The function ψ becomes zero on the curves (4.8) which lie in the region $\Phi > 0$. A repulsion zone is formed. The form of the solutions is shown at Fig. 7.

The integral curves of the system (4.3) in Fig. 5, 6, 7 are drawn in the following manner. We choose an arbitrary point P and draw the vector $\boldsymbol{\rho}$. The rotation of $\boldsymbol{\rho}$ is counterclockwise in regions where ψ is positive. In regions where ψ is negative, $\boldsymbol{\rho}$ rotates in the clockwise sense. Drawing the integral curves in regions where Φ is negative, the distance of P from the origin is reduced and in regions, where Φ is positive it is increased. The form of the integral curves in the (—) vicinity of repulsion zones and attraction is obvious.

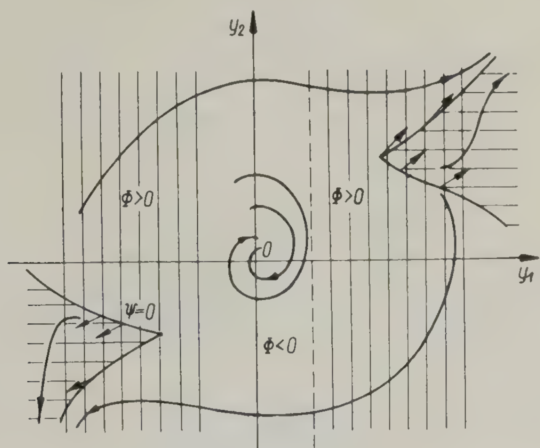

Fig. 7

From the integral curves of Figs. 5, 6, 7, it follows that the equilibrium is asymptotically stable. The asymptotic stability in Fig. 5 and 6 takes place in the entire plane and in Fig. 7 at some places only. The form of the solutions of the system (4.2) is analogous, on the basis of the theorems 1, 2 and 3 above. The present example has been solved by the method described. However, the passage from the system (4.2) to (4.3) can be avoided, the system (4.2) being written thus:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 - x_2(a+bx_2+cx_2^2). \tag{4.10}$$

This system is of the type (4.3), and can be treated directly by the method described. The passage from the system of equations (3.1) to (3.2) is necessary only when (3.1) cannot be written in the form (3.2).

Example 2. Let us apply our method to a self-excited system governed by the Van der Pol equation:

$$\ddot{y} - \varepsilon(1-y^2)\dot{y} + y = 0, \quad \varepsilon > 0. \tag{4.11}$$

Substituting $y = y_1$, $\dot{y} = y_2$, we obtain the system of equations

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = \varepsilon(1-y_1^2)y_2 - y_1. \tag{4.12}$$

The system (4.12) is of the type (3.2) and there is no need to use the transformation from the assumption 4. Let us calculate the function Φ:

$$\Phi = \varepsilon y_2^2(1-y_1^2). \tag{4.13}$$

The function Φ is non-negative in the zone $-1 \leqslant y_1 \leqslant 1$ and y_2 is arbitrary. Since in this region lies a singular point, the equilibrium is unstable. Let us investigate the non-zero coordinate of the tensor $\boldsymbol{\psi}$:

$$\psi = -y_1^2 + y_1 y_2 \varepsilon(1-y_1^2) - y_2^2. \tag{4.14}$$

The function ψ becomes zero on the curves described by the equations

$$y_2 - \frac{\varepsilon y_1}{2}(1-y_1^2) \pm \frac{y_1}{2}\sqrt{\varepsilon^2(1-y_1^2)^2-4} = 0. \tag{4.15}$$

The curves (4.15) lie in the zones determined by the inequalities

$$y_1 > \sqrt{1+\frac{2}{\varepsilon}}, \quad \text{or} \quad y_1 < -\sqrt{1+\frac{2}{\varepsilon}}, \quad \text{any } y_2 \tag{4.16}$$

$$-\sqrt{1-\frac{2}{\varepsilon}} < y_1 < \sqrt{1-\frac{2}{\varepsilon}}, \quad \text{any } y_2. \tag{4.17}$$

There is an attraction zone along the curves (4.15), lying in the zone (4.16) and a repulsion zone along those in the zone (4.17). The form of the solutions is shown in Fig. 8.

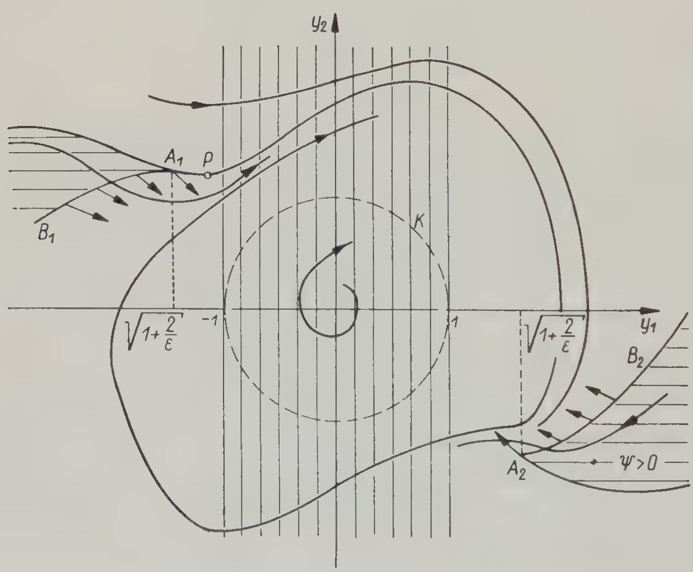

Fig. 8

From Fig. 8 it follows that the trajectory starting from the point P_0 bounds a region into which every trajectory enters. On the other hand, the circle K bounds the region from which every trajectory exits. In this way the region of the intermediate process is determined. In the case where ε is less than 2 the zone (4.17) does not exist and the curves (4.15) are the only remaining in the zone (4.16).

Example 3. A centrifugal governor of direct action, with non-linear damping, Fig. 9.

The equation of motion of the governor and the engine can be written thus.

(4.18) $T_a\dot{\varphi} = -\mu$ — equation of motion of the engine,

$T_r^2\ddot{\mu}+T_k\dot{\mu}+k\dot{\mu}^2+h\dot{\mu}^3+\delta\mu = \varphi$ — equation of motion of the governor.

T_a — acceleration period of the engine, T_r — natural vibration period of the governor multiplied by 2π, T_k — time constant of the damper, k, h – the parameters of the nonlinear characteristic of the viscosity damper, δ — the sensitivity factor or the governor, φ — variation of angular velocity, μ — the displacement of the governor sheath.

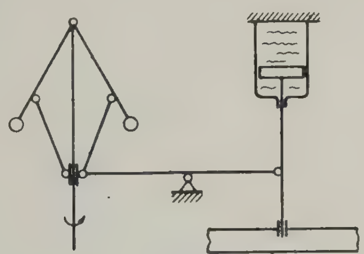

Fig. 9

Let us introduce the notations

$$a = \frac{k}{T_r^2}, \quad \beta = \frac{h}{T_k^3}, \quad A = \delta\sqrt[3]{\frac{T_a^2}{T_r^2}}, \quad B = \frac{T_k}{T_r}\sqrt{\frac{T_a}{T_r}},$$

and the transformation

$$\tau = \frac{t}{\sqrt[3]{T_a T_r^2}}, \quad x = \mu\sqrt[3]{\frac{T_r^2}{T_a^2}}.$$

With these notations, and applying the transformation the system (4.18) takes the form

$$(4.19) \qquad \dot{\varphi} = -x, \quad \ddot{x} = \varphi - Ax - B\dot{x} - aB^2\dot{x}^2 - \beta B^3\dot{x}^3.$$

Substituting $\dot{x} = x_1$, $x = x_2$, $\varphi = x_3$ we obtain a system of three equations of the first order:

$$(4.20) \qquad \begin{cases} \dot{x}_1 = -(B+aB^2x_1+\beta B^3x_1^2)x_1 - Ax_2 + x_3, \\ \dot{x}_2 = x_1, \quad \dot{x}_3 = -x_2. \end{cases}$$

The system (4.20) has been written in the form (3.2). The investigation of the behaviour of the solutions will be undertaken assuming:

$$(4.21) \qquad B^2 > 4A, \quad a^2 - 4\beta < 0, \quad \beta > 0.$$

Let us denote the quadratic trinomial in the first equation by

$$W = B + \alpha B^2 x_1 + \beta B^3 x_1^2. \tag{4.22}$$

With the assumption (4.21), the trinomial W is always positive. Let us investigate the motion of the projection of the point P moving along the trajectory, on the (x_1, x_2) plane, (Fig. 10).

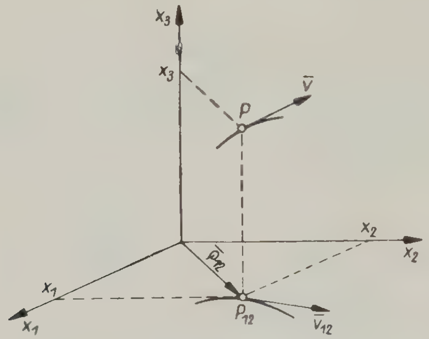

Fig. 10

The scalar product of the vectors of velocity of the projection of $\mathbf{V}_{12}$, and the distance $\boldsymbol{\rho}_{12}$, is determined by the function Φ_{12}, and the vector product of these vectors by the coordinate of the tensor $\boldsymbol{\psi}$:

$$\begin{cases} \Phi_{12} = -Wx_1^2 - x_1x_2(A-1) + x_1x_3, \\ \psi_{12} = x_1^2 + Wx_1x_2 + Ax - x_2x_3. \end{cases} \tag{4.23}$$

For every x_3 the function Φ_{12} is zero on the lines

$$x_1 = 0, \qquad Wx_1 + x_2(A-1) - x_3 = 0. \tag{4.24}$$

The regions in Fig. 11 where the function Φ_{12} is positive are shaded vertically. Let us reduce the function ψ_{12} to the sum of the squares

$$\psi_{12} = \left(x_1 + \frac{Wx_2}{2}\right)^2 + \frac{4A - W^2}{4}\left(x_2 - \frac{2x_3}{4A - W^2}\right)^2 - \frac{x_3^2}{4A - W^2}. \tag{4.25}$$

From the assumption (4.21), it follows that $W^2 > 4A$. Setting ψ_{12} equal to zero we obtain a non-parameter family of curves in the (x_1, x_2)-plane. The variable x_3 is considered to be a parameter. The behaviour of these curves for a positive fixed x_3 is represented in Fig. 11 a and for a negative x_3 in Fig. 11*b*. Attraction and repulsion zones are formed along the curves $\psi_{12} = 0$. The regions where ψ_{12} is negative are shaded horizontally. The projection of every trajectory on a plane parallel to the (x_1, x_2)-plane, tends to the point Q with the coordinates $\left(0, \frac{x_3}{A}\, x_3\right)$.

At the point Q, the trajectories intersect at right angles the planes parallel to the (x_1, x_2)-plane. Let us investigate the motion of the point Q. To this end let us pro-

ject the point P lying on the trajectory on the plane normal to the (x_1, x_2)-plane — the (x_3, x_1)-plane for instance. Let us calculate the function Φ_{31}:

$$\Phi_{31} = -Wx_1^2 - Ax_1x_2 - x_2x_3. \tag{4.26}$$

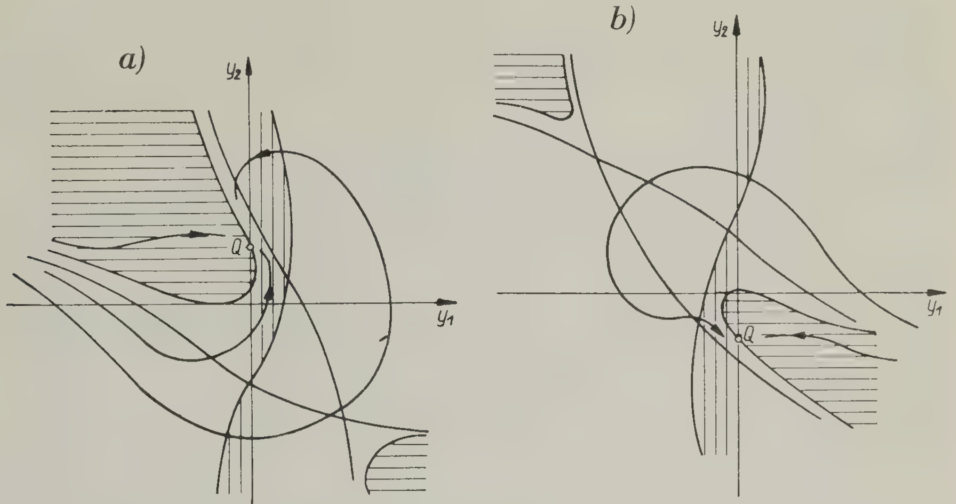

Fig. 11

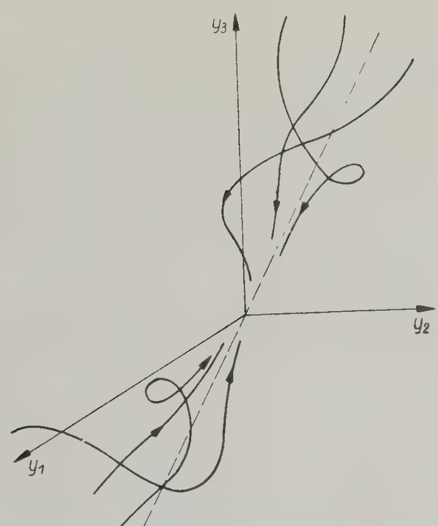

Fig. 12

If the point P lying on the trajectory tends to Q, x_1 tends to zero, x_2 tends to x_3/A, and the function Φ_{31} tends to $-x_3^2/A$. This means that the distance of Q from the origin decreases and x_3 tends to zero. The form of the solutions of the system (4.20) is represented at Fig. 12. The equilibrium is asymptotically stable in the entire space.

References

[1] Н. Д. Моисеев, *Очерки развития теории устойчивости*, Москва 1949, 415.

[2] А. М. Лапунов, *Общая задача об устойчивости движения*, Москва 1935.

[3] T. Ważewski, *Sur un principe topologique de l'examen de l'allure asymptotique de integralle des equations differentielles ordinaires*, Ann, Soc. Polon. Math., **20**, 1947.

[4] В. Я. Зубов, *Математические методы исследования систем автоматического регулирования*, Ленинград 1959.

[5] Н. Г. Четаев, *Устойчивость движения*, Москва 1955.

[6] St. Gołąb, *Rachunek tensorowy*, Warszawa 1956.

[7] Я. Я. Красовский, *Некоторые задачи теории устойчивости движения*, Москва 1959,

[8] И. И. Боголюбов, Я. А. Митропольский, *Асимпотические методы в теории нелинейных колебаний*, Москва 1958.

[9] В. В. Немыцкий, В. В. Степанов, *Качественная теория дифференциальнх уравнений*, Москва 1947.

[10] O. Joshizawa, *On the stability of solutions of the differential equation*, Mem. Col. Sci. Kyoto Univ. No 1, 1955.

[11] И. Я. Баутин, *Поведение динамических систем вблизи границ области устойчивости*, Москва 1949.

[12] А. И. Лерьер, *Введение в теорью автоматического регулирования*, Москва 1958.

[13] S. Ziemba, *Drgania swobodne o silnie nieliniowym tłumieniu*, Arch. Mech. Stos., 4, **9** (1957).

[14] W. Bogusz, *Metoda retraktowa w zastosowaniu do problemów nieliniowych w technice*, Arch. Mech. Stos., 4, **12** (1960)

[15] W. Bogusz, *Determination of stability regions of dynamic non linear systems*, Arch. Mech· Stos., 6, **11** (1959).

[16] А. Барбашин, *О динамических системах, обладающих потенциалом скоростей*, Д. А· Н,III,2, 61(1948).

[17] А. А. Дородницын, *Асимптотические решения уравнения Van der Pola*, Прикл. Мат. Мех., **11**(1947).

[18] J. Skowroński, S. Ziemba, *Some complementary remarks on the delta metod for determining phase trajectories of systems with strong non-linearity*, Arch. Mech. Stos., 5, **10** (1958).

[19] J. N. Macduff, J. R. Curreri, *Vibration Control*, New York 1958.

Streszczenie

DWUTENSOROWA METODA BADANIA UKŁADÓW NIELINIOWYCH

W pracy opisana jest metoda badania przebiegu rozwiązań układów nieliniowych autonomicznych w przestrzeni fazowej. Wykazano, że przebieg rozwiązań zależy od własności dwóch tensorów, z których jeden powstaje z iloczynu skalarnego wektorów prędkości i odległości punktu w przestrzeni fazowej, a drugi jest biwektorem prostym tych wektorów.

Metoda może być zastosowana do wyznaczania obszarów stateczności i do badania charakteru punktu osobliwego. W pracy objaśniono metodę na trzech przykładach.

Резюме

ДВУХТЕНЗОРНЫЙ МЕТОД ИССЛЕДОВАНИЯ НЕЛИНЕЙНУХ СИСТЕМ

Дается метод исследования процесса решений нелинейных автономных систем в фазовом пространстве. Доказывается, что процесс решений зависит от свойств двух тензоров, из которых первый является произведением скалярного вектора скорости и расстояния точки в фазовом пространстве, а второй простым бивектором этих векторов.

Этот метод может быть применен для определения областей устойчивости и для исследования характера особой точки. Приведенный метод иллюстрируется тремя примерами.

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NONLINEAR THEORY OF A GENERATOR WITH A LONG LINE EXCITED BY A NEGATIVE RESISTANCE

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1. Introduction

It is known that the amplitude of vibration in generating systems depends on their nonlinear properties. At the same time, due to the non-linearity of the system, the frequency of the generator depends on the amplitude. In practice, nonlinear influences on the vibration frequency can often be disregarded in relation to the linear factors (thermal, structural etc). However, even in this case the accurate knowledge of the nonlinear effects is essential, for instance, for the problem of amplitude stability. As regards generators with a long line, in spite of the fact that they are widely used in engineering — micro-wave engineering in particular the literature devoted to appropriate nonlinear phenomena is very scarce. Of importance is the paper by GROSZKOWSKI, published in 1938, [1], in which the method of harmonics proposed by that author is applied to a generator constituted by a negative resistance exciting a long line. In particular, it was shown by GROSZKOWSKI that in a generator with a long line the influence of nonlinear effects on the generator frequency may be considerably smaller than in generators with circuits with concentrated constants. The work by W. MAJEWSKI, [4], should also be mentioned, in which a long line generator is considered, the method of differential equations being used(1).

The nonlinear theory of a longline generator presented in the present paper is based on the method of the third harmonic proposed by J. GROSZKOWSKI, [2], and used also in the former paper of the author, [3]. This theory should be considered, in principle, to be qualitative, explaining the mechanism of the physical phenomena, although quantitative results can also be obtained in certain particular cases.

(1) This paper was published in 1938 in limited number of copies and is little known to the younger generation of scientific workers.

2. Admittance of the Negative Resistance

Let us consider the generating system shown at Fig. 1. This system is composed of a long line, short circuited at the end, to the input terminals of which a negative resistance G_l is connected. The long line is characterized by its wave resistance Z_0, propagation coefficient $\gamma=\alpha+j\beta$, and length l. It is assumed that the negative resistance has a nonlinear characteristic described by the equation

$$i=-S_1u+S_3u^3, \tag{2.1}$$

where i is the current and u — the voltage on the terminals of the negative resistance S_1 and S_3 are positive coefficients. Using the method of the third harmonic

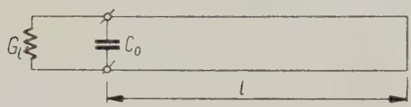


Fig. 1. The scheme of a generator with a long line excited by a negative resistance G_l

[2], [3], it is assumed that the voltage on the terminals of the negative resistance has the form:

$$u=\bar{U}_1\sin\omega t+\bar{U}_3\sin(3\omega t+\varphi_3). \tag{2.2}$$

It is known that in this case the first harmonic of the current flowing across the negative resistance is expressed by the following approximate relation:

$$i_1\approx\left(\frac{3}{4}S_3\bar{U}^3-S_1\bar{U}-\frac{3}{4}m_3S_3\bar{U}^3\cos\varphi_3\right)\sin\omega t-\frac{3}{4}m_3S_3\bar{U}^3\sin\varphi_3\cos\omega t, \tag{2.3}$$

where

$$m_3=\frac{\bar{U}_3}{\bar{U}_1} \tag{2.4}$$

is the voltage third harmonic content.

It can easily be seen that the amplitude of the current component in phase with the first harmonic of the voltage is

$$\bar{I}_1'=\frac{3}{4}S_3\bar{U}^3-S_1\bar{U}-\frac{3}{4}m_3S_3\bar{U}^3\cos\varphi_3, \tag{2.5}$$

and the amplitude of the orthogonal component is

$$\bar{I}_1''=\frac{3}{4}m_3S_3\bar{U}^3\sin\varphi_3. \tag{2.6}$$

In complex notation, we have:

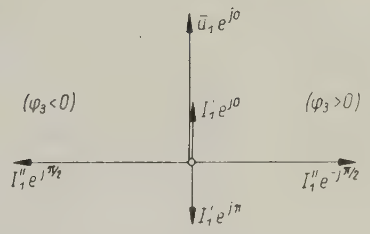
$$I_1=\bar{I}_1'e^{j0}+\bar{I}_1''^{\pm j(\pi/2)}. \tag{2.7}$$

In our considerations we shall use the notion of the admittance of the negative resistance, defined thus:

$$Y_l=\frac{I_1}{U_1}=G_l+jB_l. \tag{2.8}$$

Fig. 2 shows the phasor diagram for the first harmonic of the current and the voltage. The positive sign of the exponent $j(\pi/2)$ is used for $\varphi_3 < 0$. In this case, the negative (dynatron) resistance has a capacitive susceptance. For $\varphi_3 > 0$ the minus sign is used and the susceptance is of the inductive type.

Fig. 2. The phasor diagram for the first harmonics of voltage and current



The components G_l and B_l of the admittance are expressed by the approximate equations:

$$G_l = \frac{\bar{I}_1'}{U_1} = \frac{3}{4} S_3 \bar{U}_1^2 (1 - m_3 \cos\varphi_3) - S_1, \tag{2.9}$$

$$B_l = \frac{\bar{I}_1''}{U_1} = -\frac{3}{4} m_3 S_3 \bar{U}_1^2 \sin\varphi_3. \tag{2.10}$$

3. The Admittance of the Long Line

It is known that the input admittance of a long line short circuited at the end is determined by the equation

$$Z_w = Z_0 \frac{al \cos\beta l + j \sin\beta l}{\cos\beta l + jal \sin\beta l}, \tag{3.1}$$

where Z_0 — wave resistance, a — damping coefficient, β — propagation coefficient.

If the damping is not too strong ($al \ll 1$), the following relations hold

$$Z_0 = \sqrt{\frac{L}{C}}\left[1 + j\left(\frac{G}{2\omega C} - \frac{R}{2\omega L}\right)\right], \tag{3.2}$$

$$\beta = \omega\sqrt{LC}\left[1 - \frac{RG}{4\omega^2 LC} + \frac{G^2}{8\omega^2 C^2} + \frac{R^2}{8\omega^2 L^2}\right], \tag{3.3}$$

$$a = \frac{R}{2Z_0} + \frac{GZ_0}{2}, \tag{3.4}$$

where R, G, L, C are the constants of the line with values expressed in suitable units per unit length.

In order to avoid complicated expressions the following simplified equation will be used:

$$Z_0 \approx \sqrt{\frac{L}{C}}, \tag{3.5}$$

$$\beta \approx \omega\sqrt{LC}. \tag{3.6}$$

The components of the input impedance of the line

$$Z_w = R_w + jX_w \tag{3.7}$$

are expressed by the equations:

$$R_w = Z_0 \frac{al}{\cos^2 \beta l + a^2 l^2 \sin^2 \beta l}, \tag{3.8}$$

$$X_w = Z_0 \frac{\frac{1}{2} \sin 2\beta l (1 - a^2 l^2)}{\cos^2 \beta l + a^2 l^2 \sin^2 \beta l}. \tag{3.9}$$

The components of the input admittance

$$Y_w = \frac{1}{Z_w} = G_w + jB_w \tag{3.10}$$

are expressed by the equations

$$G_w = \frac{1}{Z_0} \frac{al}{a^2 l^2 \cos^2 \beta l + \sin^2 \beta l}, \tag{3.11}$$

$$B_w = -\frac{1}{2Z_0} \frac{\sin 2\beta l (1 - a^2 l^2)}{a^2 l^2 \cos^2 \beta l + \sin^2 \beta l}. \tag{3.12}$$

If a shunt fixed capacitance C_0 is connected to the input terminals the resulting susceptance is

$$B_w = \omega C_0 - \frac{1}{2Z_0} \frac{\sin 2\beta l (1 - a^2 l^2)}{a^2 l^2 \cos^2 \beta l + \sin^2 \beta l}. \tag{3.13}$$

4. The Admittance Equation

The steady state vibration parameters may be determined from the admittance equation:

$$Y_l + Y_w = 0. \tag{4.1}$$

This is a complex equation, that is it can be split up into two equations, the conductance equation

$$G_l + G_w = 0, \tag{4.2}$$

and the susceptance equation

$$B_l + B_w = 0. \tag{4.3}$$

5. The Conductance Equation. Determination of the Amplitude

Substituting in (4.2) the expressions (2.9) and (3.11), we obtain the conductance equation in the form

$$\frac{3}{4} S_3\bar{U}_1^2(1-m_3\cos\varphi_3)-S_1+\frac{1}{Z_0}\,\frac{al}{\alpha^2l^2\cos^2\beta l+\sin^2\beta l}=0. \tag{5.1}$$

From this equation the vibration amplitude can be determined, expressed by the equation:

$$\bar{U}_1=\sqrt{\frac{S_1-\dfrac{1}{Z_0}\,\dfrac{al}{a^2l^2\cos^2\beta l+\sin^2\beta l}}{\dfrac{3}{4}S_3(1-m_3\cos\varphi_3)}}. \tag{5.2}$$

Usually $al \ll 1$, $\beta l\approx \pi/2$ and $m_3\cos\varphi_3 \ll 1$. In this case, the vibration amplitude is expressed by the approximate formula:

$$\bar{U}_1\approx 2\sqrt{\frac{S_1-al/Z_0}{3S_3}}, \tag{5.3}$$

This is the same equation as in the case of a generator with a parallel resonance circuit having dynamic resistance R_d, because the equation

$$\frac{al}{Z_0}=\frac{1}{R_d} \tag{5.4}$$

determines the dynamic resistance of a short circuited long line for the quarter wave resonance.

Introducing the notion of the coefficient of the negative damping defined by the equation

$$\varepsilon_0=\left(S_1-\frac{al}{Z_0}\right)Z_0=S_1Z_0-al, \tag{5.5}$$

the equation for the amplitude becomes:

$$\bar{U}_1\approx 2\sqrt{\frac{\varepsilon_0}{3S_3Z_0}}. \tag{5.6}$$

6. The Susceptance Equation. Determination of the Frequency

Substituting in the Eq. (4.3) the expressions (2.10) and (3.19), we obtain the susceptance equation in the form:

$$-\frac{3}{4}m_3S_3\bar{U}^2\sin\varphi_3-\frac{1}{2\,Z_0}\,\frac{\sin 2\,\beta l(1-\alpha^2l^2)}{\alpha^2l^2\cos^2\beta l+\sin^2\beta l}+\omega C_0=0. \tag{6.1}$$

Substituting suitable equations and rearranging as shown in the Appendix 1, this equation takes the normalized form:

$$\text{(6.2)}\quad \frac{-\frac{4}{3}\varepsilon^2 \sin\left[\operatorname{arc\,tg}\frac{\frac{1}{2}\sin 6y\frac{\pi}{2}(1-a^2l^2)-3yk\frac{\pi}{2}\left(a^2l^2\cos^2 3y\frac{\pi}{2}+\sin^2 3y\frac{\pi}{2}\right)}{al}\right]}{\sqrt{\frac{a^2l^2}{\left(a^2l^2\cos^2 3y\frac{\pi}{2}+\sin^2 3y\frac{\pi}{2}\right)^2}+\left[3yk-\frac{1}{2}\,\frac{\sin 3y\pi(1-a^2l^2)}{a^2l^2\cos^2 3y\frac{\pi}{2}+\sin^2 3y\frac{\pi}{2}}\right]^2}} - \frac{1}{2}\,\frac{\sin 2y\frac{\pi}{2}(1-a^2l^2)}{a^2l^2\cos^2\frac{\pi}{2}y+\sin^2 y\frac{\pi}{2}}+yk\frac{\pi}{2}=0$$

with the following notations:

(6.3) $y=\dfrac{\omega}{\omega_0}$, ω_0—quarter wave resonance frequency;

(6.4) $k=\dfrac{C_0}{Cl}$ —the ratio of the fixed input capacitance to the total capacitance of the line;

(6.5) $\varepsilon^2=S_1Z_0-\dfrac{al}{a^2l^2\cos^2\beta l+\sin^2\beta l}$ —the coefficient of negative damping depending on the frequency.

The Eq. (6.2) enables us to determine the vibration frequency $y=\omega/\omega_0$. In view of the complicated form, the graphical method is of practical use. It is convenient to draw two curves on one diagram, one representing the dependance of the susceptance of the negative resistance on the frequency [the first, most complicated term of the Eq. (6.2)], (cf. Appendix 1, Eq. I. 17), the other — the dependance of the susceptance of the long line on the frequency, the fixed capacitance C_0 being taken into consideration [the last two terms in the Eq. (6.2)].

7. Vibration Frequency for $C_0=0$

Let us consider first the case in which the capacitance C_0 can be disregarded ($C_0\approx 0$). In this case $\varepsilon=\varepsilon_0$ [cf. Eq. (5.5)]. If now we reject a^2l^2 as small in relation to 1, and if we consider the range where $\sin^2 y(\pi/2)\gg a^2l^2\cos^2 y(\pi/2)$, the Eq. (6.2) will take the following simplified form:

$$\text{(7.1)}\quad \frac{-\frac{4}{3}\varepsilon^2\sqrt{a^2l^2+\frac{1}{4}\sin^2 3y\frac{\pi}{2}}\ \sin\left[\operatorname{arc\,tg}\frac{\sin 3y\pi}{2al}\right]}{\cos^2 3y\frac{\pi}{2}+a^2l^2\sin^2 3y\frac{\pi}{2}}-\operatorname{ctg} y\frac{\pi}{2}=0.$$

The diagram of this relation for a certain value of the parameters ε_0 and al is shown at Fig. 3. It is seen that the curve expressing the dependency of the susceptance of the negative resistance on the frequency has the same form as for the frequency discriminator. The intersection with the curve $\text{ctg}\, y(\pi/2)$ takes place at the point $y = 1$, that is the generator operates exactly with the quarter wave resonance frequency, independently of the value ε. In the ideal case under consideration there are therefore no nonlinear influences acting on the vibration frequency.

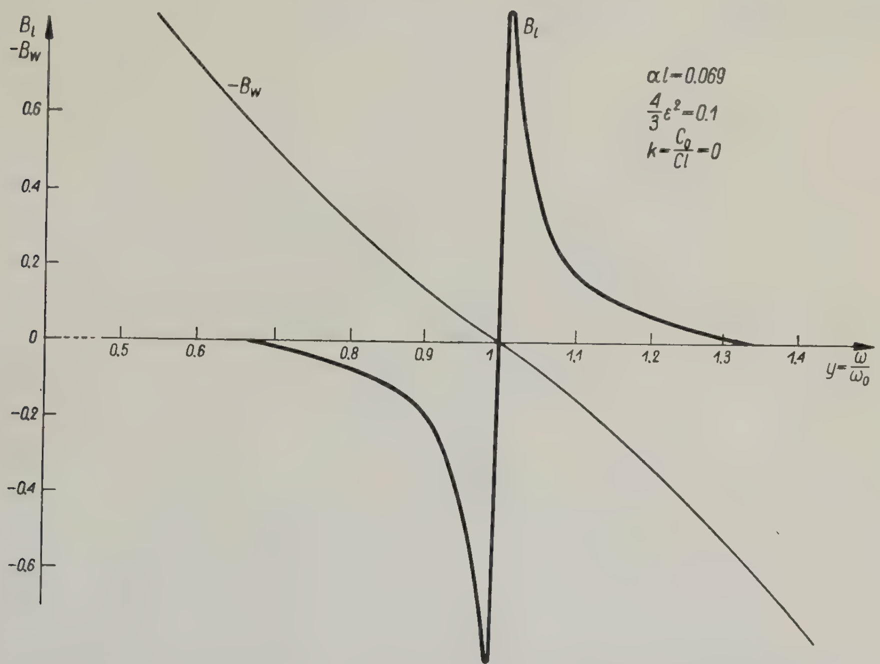


Fig. 3. The susceptance diagram of the negative resistance B_l, and the susceptance of the line B_w in function of the normalized frequency $y = \omega/\omega_0$. Note: B_w is taken with the opposite sign

This property of frequency stabilization was foretold by J. GROSZKOWSKI in an analytic way by means of the method of harmonics, [1].

8. Vibration Amplitude and Frequency, the Input Capacitance C_0 Being Taken into Account

The Eq. (7.1) and the diagram of Fig. 3 concern the ideal case of $C_0 = 0$. In practice, however, the influence of the fixed capacitance C_0 should be taken into consideration, therefore we are interested in the solution according to the Eq. (6.2). In addition to the quantity ε and the parameter al, there appears in this equation a parameter $k = C_0/Cl$ characteristic of the influence of the capacity C_0.

If nonlinear influences are rejected, the frequency is determined, according to the linear theory, by the relation:

$$yk\frac{\pi}{2} - \operatorname{ctg} y\frac{\pi}{2} = 0. \tag{8.1}$$

Fig. 4 shows a diagram of the dependency of the frequency on the parameter k. If, however, the nonlinear action of the third harmonic is taken into consideration, the vibration frequency is somewhat higher then that following from (8.1) and is determined by the Eq. (6.2). It should be added that with a frequency higher than the resonance frequency the admittance of the line is of the capacitive char-

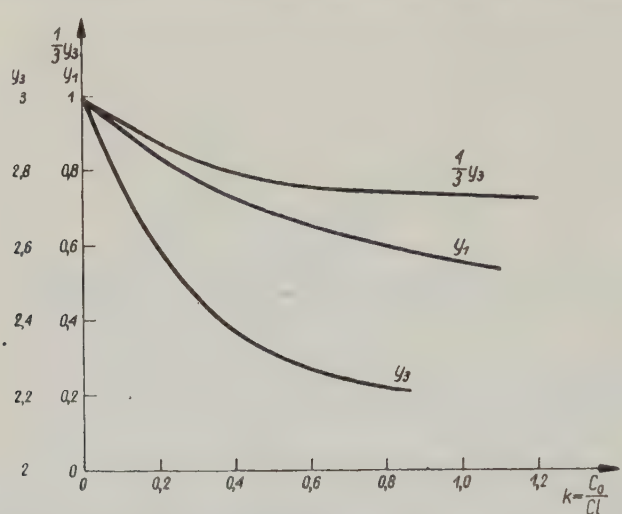


Fig. 4. The relation between the frequency and the ratio of the fixed input capacity C_0 to the total capacity of the line Cl; $y_1 = \omega/\omega_1$ is the normalized frequency of the fundamental mode, $y_3 = \omega_3/\omega_0$—the normalized frequency of the second order mode

acter and the admittance of the dynatron is of the inductive character. We can speak of the «inductance» of a dynatron in place of the notion of the «capacitance» of a dynatron introduced by Groszkowski in the case where a circuit with fixed constants, [2], is excited by a dynatron. The physical cause of the difference lies in the fact that a long line has an inductive admittance for the third harmonic, while the circuit of parallel resonance has a capacitive one.

Fig. 5 shows graphical solutions of the Eq. (6.2) for various values of the parameter k. It is shown that with increasing k, the susceptance curves of the negative resistance are shifted to the left in relation to the curve for $k = 0$. In addition they are asymmetric and show a decreasing slope in the middle portion. It is essential, that the intersection of the curves for the two susceptances takes place below the axis of abscisae, which means that the susceptance of the negative resistance is of the inductive type and that of the line of the capacitive type as explained

above. Since the scale of the curve for the susceptance of the line varies with varying ε_0, the frequency varies somewhat with varying ε_0. This variation depends on the slope of the discriminator curve in its middle portion. It is seen from the Fig. 5 that this slope decreases for large k. However, in view of the great slope of the curve, the frequency variation due to the variation of ε_0 is much smaller than in a dynatron generator with a circuit with fixed constants.

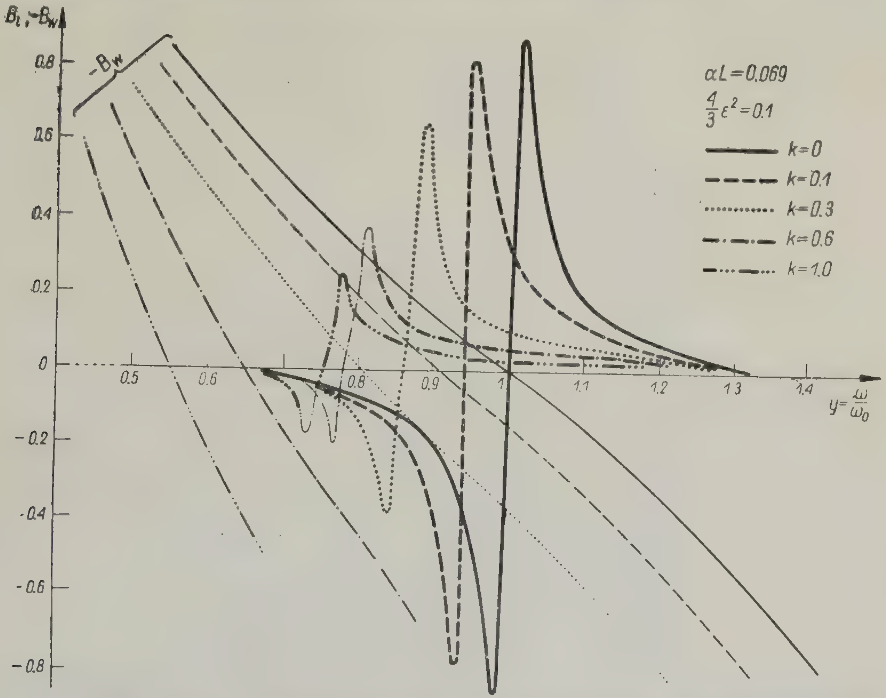


Fig. 5. The diagram of the susceptance of the negative conductivity B_1 and the susceptance of the line B_w in function of the normalized frequency $y = \omega/\omega_1$ for various $k = C_0/C_l$. Note: B_w is taken with the opposite sing.

It is characteristic that above a certain limit value k the curves of Fig. 4 do not intersect, which means that the phase condition cannot be satisfied although the negative damping coefficient ε is greater than zero. For, according to the Eqs. (5.2) and (6.5), this takes place when

$$\varepsilon = \left(S_1 Z_0 - \frac{al}{\alpha^2 l^2 \cos^2 y\frac{\pi}{2} + \sin^2 y\frac{\pi}{2}}\right) \geqslant 0. \tag{8.2}$$

If $\sin^2 y(\pi/2) \gg \alpha^2 l^2 \cos^2 y(\pi/2)$ we can write in an approximate manner

$$\varepsilon = \left(S_1 Z_0 - \frac{al}{\sin^2 y\frac{\pi}{2}}\right) \geqslant 0\,. \tag{8.3}$$

This equation can be rewritten thus

$$\varepsilon = \varepsilon_0 + al\left(1 - \frac{1}{\sin^2 y\frac{\pi}{2}}\right) \geqslant 0. \tag{8.4}$$

The relation $\varepsilon = f(y)$ is drawn as an example in Fig. 6 for $al = 0.069$. The boundary condition $\varepsilon = 0$ corresponds to the value of frequency determined by the equation:

$$\sin y\frac{\pi}{2} = \sqrt{\frac{1}{1+\frac{\varepsilon_0}{al}}}. \tag{8.5}$$

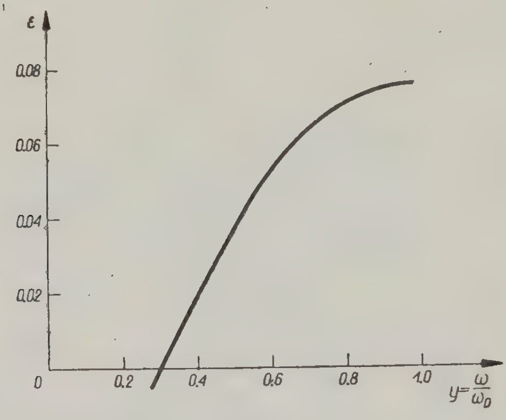


Fig. 6. The dependency of the negative damping coefficient ε on the relative frequency $y = \omega/\omega_0$. The damping $al = 0.069$, $s_1 z_1 = 0.343$, $\varepsilon_1 = 0.075$.

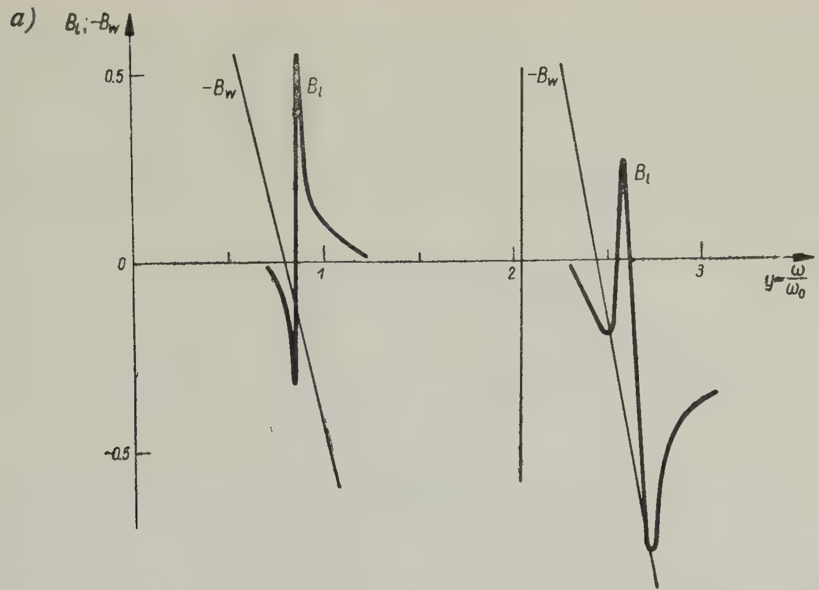
It is seen that in the concrete case represented by Fig. 6 the amplitude condition is satisfied for $y \geqslant 0.3$, while the phase condition is satisfied if $y \geqslant 0.75$ (approximately).

9. Generation of Vibration of Higher Orders

A generator with a long line has theoretically an infinite number of degrees of freedom. In the ideal case of a line of zero input capacitance and with the damping independent of the frequency, the generator can, according to the linear theory, excite itself on every frequency, for which

$$\operatorname{ctg} y\frac{\pi}{2} = 0; \tag{9.1}$$

that is, for $y = 1, 3, 5, \ldots$. The steady state frequency of the generator would be accidental. If the influence of the input capacitance is taken into consideration, the frequencies of vibrations of higher order are not in a harmonic relation and are determined by solving the Eq. (8.1). For the nonlinear effects the Eq. (6.2)



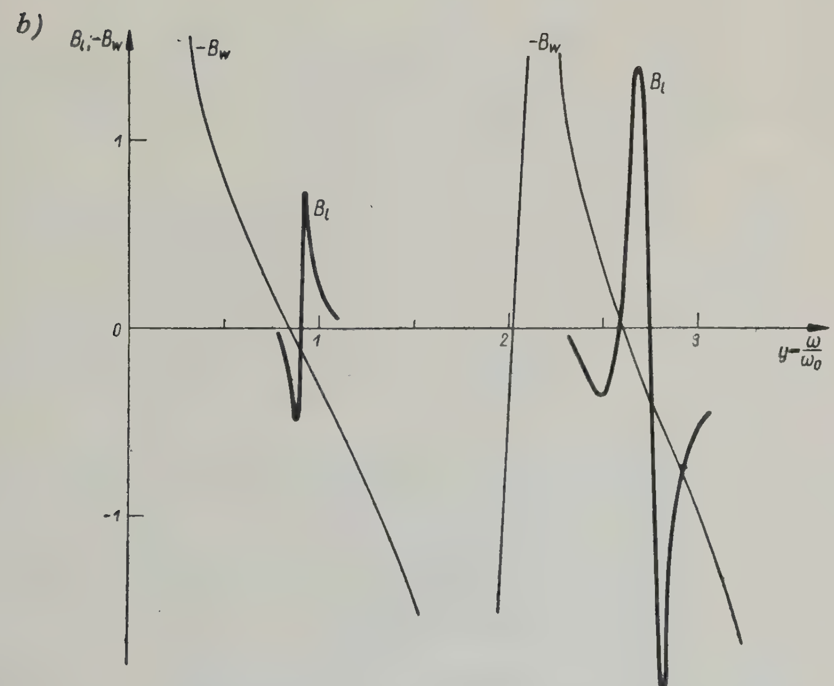


Fig. 7. Diagram of the susceptance of the negative resistance B_l and the input susceptance of the line B_w in function of the normalized frequency $y = \omega/\omega_0$ in the range of the fundamental and the second order mode: (a) $k = C_0/C_l = 0.2$; (b) $k = 0.3$.

can be used by enlarging the frequency range in an appropriate manner, for which the susceptance curves are determined. Fig. 7 shows the curves for a second order vibration for the parameters $k = 0.2$ and $k = 0.3$. It is seen that in the given case the intersection of the curves takes place at three points. The problem as to whether any of these points determines a stable of vibration remains open, however. This problem will not be considered here. We shall only point out the fact that the stability condition has the form, [3],

$$\frac{\partial G}{\partial U}\frac{\partial B}{\partial y} - \frac{\partial G}{\partial y}\frac{\partial B}{\partial U} > 0,$$

where

$$G = G_l + G_w, \quad B = B_l + B_w$$

[see Eqs. (4.1), (4.2), (4.3)].

10. Conclusions

On the basis of considerations taking nonlinear effects into consideration, the mechanism of steady-state vibration in a generator with a long line short-circuited at the end is described. In particular, it is shown that in the case under consideration the dynatron has an inductive admittance. In addition, it follows from the considerations that in the case where the input capacitance of the line is too high, the phase condition is not satisfied in spite of a negative damping of the system. Unfortunately, this fact has not yet been confirmed experimentally. The chief difficulty consists in that the ideal case of negative resistance of cubic characteristic was considered. The characteristic of the nonlinear resistance used in the experiments was asymmetric, that is it can be described by an equation containing a quadratic term. In consequence, there was an influence of the second harmonic. The investigations of the experimental layout being of a qualitative character, neither a description of the layout nor the results will be given. However, the qualitative investigations confirmed the conclusion that due to the influence of the third harmonic, the generator produces vibrations of frequency higher than the resonance frequency of the line. The influence of the second harmonic causes a decrease in the frequency of the generator, the line admittance of the second harmonic being of the capacitance type.

Generators with long lines are most commonly used in the *uhf* and the microwave range. Their electrical layouts are more complicated than that in Fig. 1. In addition the negative resistance characteristic may have, as was mentioned, a shape different from that used in the considerations. For these causes, the theory given here concerns a certain ideal case and its object is a qualitative explanation of nonlinear influences in generators with long lines. Let us observe, however, that the equivalent scheme layout of a generator with a long line excited by a tunnel diode approaches the scheme in Fig. 1. The characteristic of the negative resistance of a tunnel diode is of the dynatron type and may be approximated by the characteristic according to the Eq. (2.1). The extension of the present theory to the case

of a characteristic containing the square term is possible but the equations become very complicated.

A generator with a long line has many degrees of freedom and investigation of the possibility of self excitation of higher order vibrations is also essential. In principle, the Eq. (6.2) enables the investigation of the characteristic of the steady state also for higher order vibrations. In practice, however, the fact that the damping of the line is a function of frequency should be taken into consideration. In the experimental generator, no higher order vibration was observed either in the steady state or during build up of vibrations.

The theory presented was the subject of a lecture held by the author at the conference on non-linear vibrations organised by Czechoslovak and Polish Academy of Sciences at Liblice near Prague on September, 21-24, 1960.

Appendix

In agreement with (2.10) the susceptance of the negative resistance is expressed the equation:

$$B_l = -\frac{3}{4} m_3 S_3 \bar{U}^2 \sin \varphi_3. \tag{A. 1}$$

Since the dependency of B_l on frequency is desired, therefore we must know in which way m_3 and φ_3 vary with the frequency.

The current of the third harmonic is expressed by the approximate formula

$$i_3 \approx -\frac{1}{4} S_3 \bar{U}^3 \sin \omega t. \tag{A. 2}$$

Its amplitude is therefore equal to:

$$\bar{I}_3 = \frac{1}{4} S_3 \bar{U}^3. \tag{A. 3}$$

The content of the third harmonic is determined by the equation

$$m_3 = \frac{\bar{U}_3}{\bar{U}_1} = \frac{\bar{J}_3 |Z_{3w}|}{\bar{U}_1}, \tag{A. 4}$$

where $|Z_{3w}|$ is the modulus of the input resistance of the line for the frequency of the third harmonic, the capacitance C_0 connected in parallel, being taken into account.

The input conductance of the line is expressed by the Eq. (3.11):

$$G_{3w} = \frac{1}{Z_0} \frac{al}{a^2 l^2 \cos^2 3\beta l + \sin^2 3\beta l}. \tag{A. 5}$$

The resultant input conductance of the line, taking the capacitance C_0 into consideration, is

$$B_{3w} = 3\omega C_0 - \frac{1}{2Z_0} \frac{\sin 6\beta l(1-a^2 l^2)}{a^2 l^2 \cos^2 3\beta l + \sin^2 3\beta l}. \tag{A. 6}$$

Hence the modulus of the input impedance of the line is

(A. 7) $$|Z_{3w}| = \frac{1}{\sqrt{G_{3w}^2+B_{3w}^2}} =$$

$$= \frac{1}{\sqrt{\dfrac{1}{Z_0^2}\,\dfrac{\alpha^2 l^2}{(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l)^2}+\left[3\omega C_0-\dfrac{1}{2Z_0}\,\dfrac{\sin 6\beta l(1-\alpha^2 l^2)^2}{\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l}\right]^2}}\,.$$

Substituting (A. 7) and (A. 3) in (A. 4), we obtain:

(A. 8) $$m_3 = \frac{\frac{1}{4} S_3 \bar{U}_1^2 Z_0}{\sqrt{\dfrac{\alpha^2 l^2}{(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l)^2}+\left[3\omega C_0 Z_0-\dfrac{1}{2}\,\dfrac{\sin 6\,\beta l(1-\alpha^2 l^2)}{\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l}\right]^2}}\,.$$

The angle φ_3 will be determined by taking the relation

(A. 9) $$U_3 = -J_3 Z_{3w},$$

where

(A.10) $$Z_{3w} = R_{3w} + jX_{3w}$$

is the impedance for the third harmonic. Let us construct the phasor diagram for voltages and currents at the moment $t = 0$ (Fig. 8).

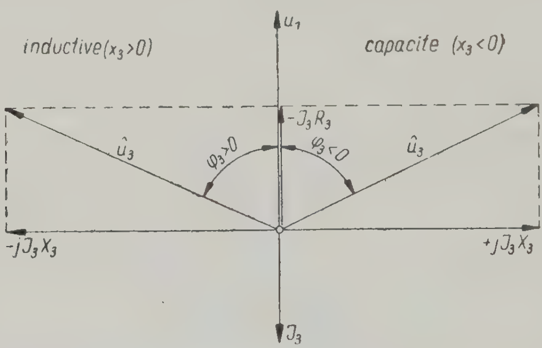


Fig. 8. The phasor diagram of voltages and currents of the fundamental and the third harmonic

It is seen from the diagram that if $X_3>0$ (the line shows the inductive properties for the frequency 3ω), than the angle φ_3 is positive and contained in the range

(A. 11) $$\pi/2 \geqslant \varphi_3 \geqslant 0.$$

If $X_3<0$ (the line shows the capacitive properties), then the angle φ_3 is negative and contained in the range

(A. 12) $$-\pi/2 \leqslant \varphi_3 \leqslant 0.$$

The angle φ_3 will be found from the equation

(A. 13)
$$\operatorname{tg}\varphi_3 = -\frac{B_{3w}}{G_3};$$

or, substituting (1.6) and (1.5), we obtain:

(A. 14)
$$\operatorname{tg}\varphi_3 = \frac{\dfrac{1}{2Z_0}\,\dfrac{\sin 6\beta l(1-\alpha^2 l^2)}{\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l}-3\omega C_0}{\dfrac{1}{Z_0}\,\dfrac{al}{\alpha^2 l^2\cos^2\beta l+\sin^2\beta l}}.$$

After some transformations, we obtain:

(A. 15)
$$\varphi_3 = \operatorname{arc\,tg}\frac{\frac{1}{2}\sin 6\beta l(1-\alpha^2 l^2)-3\omega C_0 Z_0(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l)}{al}.$$

Substituting (A. 8) and (A. 15) in (A. 1), we obtain:

(A. 16)
$$B_l = \frac{-\dfrac{3}{4}S_3^2\bar{U}_1^4 Z_0\sin\left[\operatorname{arc\,tg}\dfrac{\frac{1}{2}\sin 6\beta l(1-\alpha^2 l^2)-3\omega C_0 Z_0(\alpha^2 l^2\cos^2 3\beta l+\sin^2\beta^2 l)}{\alpha l}\right]}{\sqrt{\dfrac{\alpha^2 l^2}{(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l)^2}+\left[3\omega C_0 Z_0-\dfrac{1}{2}\,\dfrac{\sin 6\beta l(1-\alpha^2 l^2)}{\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l}\right]^2}}.$$

Substituting in this equation the relation (5.6), we obtain:

(A. 17)
$$B_l = \frac{-\dfrac{4}{3}\varepsilon^2\dfrac{1}{Z^0}\sin\left[\operatorname{arc\,tg}\dfrac{\frac{1}{2}\sin 6\beta l(1-\alpha^2 l^2)-3\omega C_0 Z_0(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l)}{\alpha l}\right]}{\dfrac{\alpha^2 l^2}{(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l)^2}+\left[3\omega C_0 Z_0-\dfrac{1}{2}\,\dfrac{\sin 6\beta l(1-\alpha^2 l^2)}{\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l}\right]^2}.$$

The susceptance equation (4.3) and (6.1) takes, on substituting (A. 17), the form:

(A. 18)
$$\frac{-\dfrac{4}{3}\varepsilon^2\sin\left[\operatorname{arc\,tg}\dfrac{\frac{1}{2}\sin 6\beta l(1-\alpha^2 l^2)-3\omega C_0 Z_0(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l}{al}\right]}{\dfrac{\alpha^2 l^2}{(\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l)^2}+\left[3\omega C_0 Z_0-\dfrac{1}{2}\,\dfrac{\sin 6\beta l(1-\alpha^2 l^2)}{\alpha^2 l^2\cos^2 3\beta l+\sin^2 3\beta l}\right]^2}-$$
$$-\frac{1}{2}\,\frac{\sin 2\beta l\,(1-\alpha^2 l^2)}{\alpha^2 l^2\cos^2\beta l+\sin^2\beta l}+\omega C_0 Z_0=0.$$

This equation may be reduced to the normalized form by denoting:

$$\beta l=\frac{\omega}{\omega_0}\,\frac{\pi}{2}=y\frac{\pi}{2}, \qquad \text{where}\quad y=\frac{\omega}{\omega_0}, \qquad \omega_0=\frac{\pi}{2\sqrt{LC}\,l},$$

$$\omega C_0 Z_0=\frac{\omega}{\omega_0}\,\omega_0 C_0 Z_0=yk\frac{\pi}{2}, \qquad \text{where}\quad k=\frac{C_0}{C_0 L}.$$

We have assumed $Z_0=\sqrt{L/C}$. With these notations, we have:

(A. 19)
$$\frac{-\frac{4}{3}\varepsilon^2 \sin\left[\operatorname{arc\,tg}\frac{\frac{1}{2}\sin 3y\pi(1-\alpha^2l^2)-3yk\left(\alpha^2l^2\cos^2 3y\frac{\pi}{2}+\sin^2 3y\frac{\pi}{2}\right)}{\alpha l}\right]}{\sqrt{\frac{\alpha^2l^2}{\left(\alpha^2l^2\cos^2 3y\frac{\pi}{2}+\sin^2 3y\frac{\pi}{2}\right)^2}+\left[3yk-\frac{1}{2}\frac{\sin 3y\pi(1-\alpha^2l^2)}{\alpha^2l^2\cos^2 3y\frac{\pi}{2}+\sin^2 3y\frac{\pi}{2}}\right]^2}}-$$
$$-\frac{1}{2}\frac{\sin y\pi(1-\alpha^2l^2)}{\alpha^2l^2\cos^2 y\frac{\pi}{2}+\sin^2 y\frac{\pi}{2}}+yk\frac{\pi}{2}=0.$$

For the derivation of the Eqs. (A.17) to (A.19) it was assumed that the amplitude of vibration is expressed by the Eg. (5.6) and does not depend on βl. If (5.6) is replaced by (5.2) then in the Eq. (A. 19) the following expression should be substituted in place of the constant coefficient of negative damping

$$\varepsilon^2=\left(S_1Z_0-\frac{al}{\alpha^2l^2\cos^2\beta l+\sin^2\beta l}\right)^2.$$

References

[1] J. Groszkowski, *Stabilizacja częstotliwości przy pomocy obwodu o stałych równomiernie rozłożonych*, Kwart. Telekomunik., 1, 1938, 1-4; 2, 1938, 17-20.

[2] J. Groszkowski, *O indukcyjności łuku i pojemności dynatronu*, Arch. Eletrotechn., 1, **3** (1954), p. 3; Bull. Acad. Polon. Sci. Ser. Sci. Techn., **3**, 1955, p. 87.

[3] S. Hahn, *Zawartość harmonicznych, amplituda i częstotliwość cyklu granicznego Van der Pola, Metoda równowagi admitancji*, Arch. Elektrotechn., 3, **6** (1957), 371-394.

[4] W. Majewski, *Sur l'excitation des oscillation dans un circuit a constantes electriques uniforment réparties a L'aide d'un dynatron*, Ann. Acad. Sci. Techn. Varsovie, **5** (1938), p. 43.

Streszczenie

TEORIA NIELINIOWA GENERATORA O LINII DŁUGIEJ POBUDZANEJ OPOREM UJEMNYM

Rozpatrzono generator złożony z linii długiej ćwierćfalowej na końcu zwartej, pobudzanej do drgań oporem ujemnym dynatronowym o charakterystyce sześciennej.

Stosując metodę równania admitancji ułożono równania, z których można wyznaczyć amplitudę i częstotliwość drgań. Rozpatrzono przypadek idealny z pominięciem wpływu pojemności skupionej C_0, podłączonej równolegle do zacisków wyjściowych linii, jak również przypadek z uwzględnieniem wpływu tej pojemności.

Wykazano, że jeśli pojemność jest zbyt duża, to warunek fazy nie jest spełniony, mimo że odtłumienie generatora ε jest dodatnie.

Резюме

НЕЛИНЕЙНАЯ ТЕОРИЯ ГЕНЕРАТОРА С ДЛИННОЙ ЛИНИЕЙ ВОЗБУЖДАЕМОЙ ОТРИЦАТЕЛЬНЫМ СОПРОТИВЛЕНИЕМ

Рассматривается генератор, состоящий из коротко-замкнутой четверть-волновой линии, возбуждаемой отрицательным сопротивлением динатронного типа, с кубической характеристикой.

Применяя метод уравнения адмитанса составляются уравнения, по которым можно определить амплитуду и частоту колебаний.

Рассматривается идеальный случай при пренебрежении влиянием сосредоточенной емкости C_0, подключенной параллельно к входным зажимам линии, а также случай с учетом влияния этой емкости.

Доказывается, что если емкость слишком большая, тогда условие фазы не удовлетворено несмотря на то, что отрицательное затухание колебаний генератора ε является положительным.

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THE UNLOADING WAVE IN A LAYERED BODY WITH RIGID UNLOADING CHARACTERISTIC

SYLWESTER KALISKI (WARSAW)

1. Introduction

The problem of propagation and reflection of a loading and unloading wave from a rigid and elastic wall was solved, for a body assumed to have a rigid unloading characteristic, in Refs. [1] and [2]. In spite of the qualitative difference between this solution and the reality (in the unloading zone, the body behaves as a rigid body subjected to an initial deformation) the quantitative results very closely approach the reality. A similar problem for a free surface was treated in the paper [3]. Analogous solutions for non-homogeneous bodies with non-rigid unloading characteristic were investigated by using the technique of numerical solutions in J. OSIECKI'S doctorate thesis, [4], where the limit cases of solution for rigid unloading characteristics were discussed and where the above statements found confirmation.

As regards practical problems, we are essentially interested, however, in solutions convenient for a designer — in a closed form, if possible. In favour of this interest there is also the fact that approximate mathematical operations result also in an error, which generally increases with time if the method of nets is used, (cf. [4]), while for a physical approximation, for which an accurate mathematical solution can be obtained, the causes and the extent of error are fully known. For these reasons it appears in the author's opinion, to be worth while, to consider some other solutions, based on the assumption that the body has a rigid unloading characteristic. Since the problem of plastic waves in layered bodies is often met with in practice, a solution for such a body will be presented.

The problem of the unloading wave will be treated, that of the loading wave being well known and considered to be classical.

In subsequent papers, the problem of the unloading wave will also be solved for a cylindrical wave and a spherical wave — which requires another approach, in view of the composite state of stress.

The solution given below can easily be generalized to the case of bodies non-homogeneous in each particular layer and to cases of curvilinear or angular loading characteristic (Fig. 1).

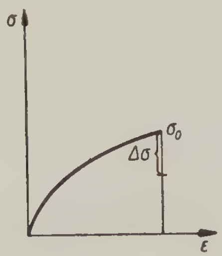

Fig. 1

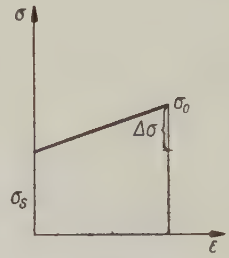

Fig. 2

However, for the sake of clarity, and in order to obtain relatively simple equations, we shall consider the cases of the $(\sigma-\varepsilon)$-characteristic represented in Figs. 2 and 3. Both cases find practical application to various bodies, in soil mechanics in particular. For moderate soil pressures, diagram 3 may sometimes be assumed as an approximation of the real diagram given in Fig. 4.

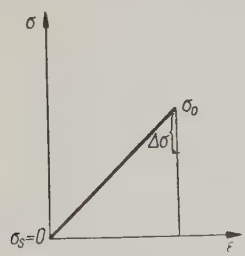

Fig. 3

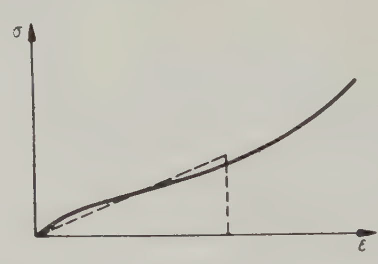

Fig. 4

The viscosity is not taken into consideration because we shall confine ourselves to pressures at the soil surface varying relatively slowly in time in the unloading range (certain explosions, for instance, for which the unloading time is of the order (0.1—1.0)/sec). It is of course an approximate assumption.

In addition, in order not to complicate the solution for reflected waves, we shall consider, similarly to [2], a strong discontinuity wave. Weak discontinuity waves involve, for reflection, certain additional complications which do not, however, change the essential features of the problem, the solution thus being constructed following the rules given below. Therefore this problem will not be treated in detail.

Let us consider first the general solution in a qualitative manner, by means of the phase plane, for the $(\sigma-\varepsilon)$-characteristics as shown in Figs. 2 and 3. Then, some simple examples will be given. As already indicated, we can construct, in principle, the solution for a $(\sigma-\varepsilon)$-characteristic as represented in Fig. 1.

The solution becomes considerably complicated, however, and loses clarity because of, among others things, the fact that the unloading wave becomes a curve, [1], even for waves of strong discontinuity. In view of the requirement of simplicity this case will be omitted.

2. The Unloading Wave in the Case $\varrho_2 a_2 > \varrho_1 a_1$

Let us consider the one-dimensional case of the semi-space to which pressure is applied suddenly on the surface, and then decreases monotonically. A two-layer medium is assumed to occupy the semi-space. In the case of a greater number of layers, the solutions will be repeated. (Then, there exists also a possibility of the appearance of weak discontinuity waves).

Let us consider three cases: (1) $\sigma_{s1} = \sigma_{s2} = 0$, (2) $|\sigma_{s1}| < |\sigma_{s2}|$, (3) $|\sigma_{s1}| > |\sigma_{s2}|$. Although the first case is a particular case of the other two, the relations in the phase plane change in this case in an essential manner, therefore it will be treated separately. Let us start from this case. The $(\sigma - \varepsilon)$-diagram is that of Fig. 2.

Case 1. $\sigma_{s1} = \sigma_{s2} = 0$. Since various sub-cases may be considered for various durations of the pressure on the surface, various distances of the second layer etc., therefore only one of them will be considered, other variants which can be solved

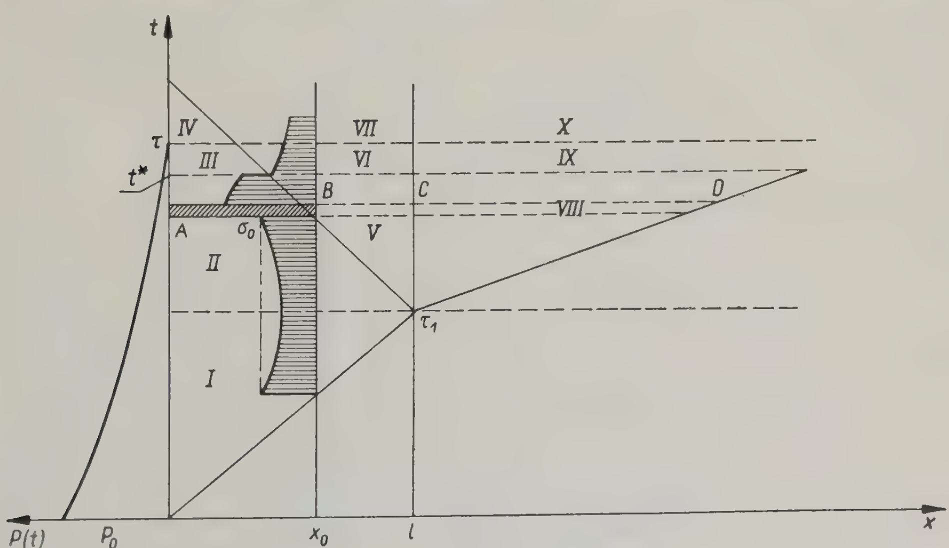

Fig. 5

in the same manner being disregarded. The parameters will be assumed as shown in Fig. 5. Each particular solution zone is denoted by a different number in Fig. 5. The relevant solutions will be denoted by the same labels.

In order to fix the attention, $a_2 > a_1$ has been assumed in Fig. 5, which corresponds to the condition $\varrho_2 a_2 > \varrho_1 a_1$ for $\varrho_1 = \varrho_2$.

Let us establish the equations and obtain the solution for each particular zone. We assume that $P(t)$ is positive for compression.

Zone I. In agreement with the results of Ref. [1], we have:

$$v_1 = \frac{1}{\varrho_1 a_1 t} H(t) = v_{1p}, \tag{2.1}$$

$$\varepsilon_1 = -\frac{a_1}{E_1 x} H\left(\frac{x}{a_1}\right) = \varepsilon_{1p}, \tag{2.2}$$

$$\sigma_1 = \int_0^x \frac{1}{x_1}\left[P\left(\frac{x_1}{a_1}\right) - \frac{a_1}{x_1} H\left(\frac{x_1}{a_1}\right)\right] dx_1 - \\ - \frac{a_1}{x} H\left(\frac{x}{a_1}\right) + \frac{x}{a_1 t^2} H(t) + P_0 - \left(1 - \frac{x}{a_1 t}\right) P(t), \tag{2.3}$$

$$\sigma_{1p} = -\frac{a_1}{x} H\left(\frac{x}{a_1}\right), \tag{2.4}$$

where

$$H(t) = \int_0^t P(t_1) dt_1, \tag{2.5}$$

and the subscript p denotes the values of the functions at the front of the wave.

Zone II. The equation of motion in the Zone II is

$$\varrho_1 (2l - a_1 t) \frac{dv_2}{dt} = P(t) + \sigma_{1p}, \tag{2.6}$$

with the initial condition

$$v_2 = \frac{1}{\varrho_1 a_1 \tau_1} H(\tau_1) = v_1(\tau_1) \quad \text{for} \quad t = \tau_1. \tag{2.7}$$

In addition,

$$\varepsilon_2 = \varepsilon_{1p}. \tag{2.8}$$

The stress $\sigma_2(x, t)$ can be expressed thus

$$\sigma_2(x, t) = \sigma_{1p}(x) - \Delta\sigma_2(x, t), \tag{2.9}$$

where the symbol $\Delta\sigma$ is explained by Fig. 3.

For an element dx of a bar AB, in the Zone II of the phase plane (Fig. 5), we can write the equilibrium equation

$$\varrho_1 \frac{\partial v_2}{\partial t} - \frac{\partial \sigma_2}{\partial x} = 0, \tag{2.10}$$

or

$$\frac{\partial \Delta\sigma_2}{\partial x} = \frac{\partial \sigma_{1p}}{\partial x} - \varrho_1 \frac{\partial v_2}{\partial t} \tag{2.11}$$

with the boundary condition

$$\Delta\sigma_2 = \psi(t) \qquad \text{for} \quad x = 0_2 \tag{2.12}$$

where $\psi(t) = P(t) - P_0$.

Hence

$$\Delta\sigma_2 = \int_0^x \left(\frac{\partial\sigma_{1p}}{\partial x} - \varrho_1\frac{\partial v_2}{\partial t}\right) dx + \psi(t). \tag{2.13}$$

Thus all the quantities of interest have been determined for Zone II.

Zone V and VIII. In the zones V and VIII the velocities of bars of unit cross-sectional area separated mentally from the semi-space should be equal; therefore

$$v_5 = v_8. \tag{2.14}$$

The stress excess σ_0 at the front of the reflected wave is

$$\sigma_0 = -\varrho_1 a_1(v_2 - v_5) \tag{2.15}$$

and

$$\sigma_{2p} = \sigma_8 = \sigma_{1p} + \sigma_0. \tag{2.16}$$

The equations of motion determining v_5 and v_8 [for v_2 determined from the Eq. (2.6)] have the form:

$$\begin{cases} \varrho_1(a_1 t - l)\dfrac{dv_5}{dt} = -(\sigma_0 + \sigma_{1p}) + \sigma_l, \\ \varrho_2[a_2(t-\tau_1)]\dfrac{dv_8}{dt} = -\sigma_l + \sigma_{8p} = -\sigma_l - \varrho_2 a_2 v_8, \end{cases} \tag{2.17}$$

or

$$[(\varrho_1 a_1 + \varrho_2 a_2)t - (\varrho_1 l + \varrho_2 a_2 \tau_1)]\frac{dv_5}{dt} + (\varrho_1 a_1 + \varrho_2 a_2)v_5 = -\sigma_{1p} + \varrho_1 a_1 v_2, \tag{2.18}$$

with the initial condition

$$v_5(\tau_1) = 2v_1(\tau_1)\frac{\varrho_1 a_1}{\varrho_1 a_1 + \varrho_2 a_2}. \tag{2.19}$$

For $\varrho_1 = \varrho_2 = \varrho$ the Eqs. (2.18) and (2.19) take the form:

$$\varrho[(a_1 + a_2)t - (l + a_2\tau_1)]\frac{dv_5}{dt} + \varrho(a_1 + a_2)v_5 = -\sigma_{1p} + \varrho a_1 v_2\ , \tag{2.20}$$

$$v_5(\tau_1) = 2v_1(\tau_1)\frac{a_1}{a_1 + a_2}. \tag{2.21}$$

Next, we shall have of course:

$$\varepsilon_5 = \frac{\sigma_8}{\varrho_1 a_1^2}, \qquad \varepsilon_8 = \varepsilon_{8p}. \tag{2.22}$$

σ_5 and σ_8 are computed in the same manner as in the zone II, except that for the determination of the function $\psi(t)$ we have now the conditions:

$$(2.23)\qquad \begin{cases} \sigma_5(x,t) = \sigma_{1p}(x,t)+\sigma_0(x)-\Delta\sigma_5(x,t), \\ \Delta\sigma_5 = 0 \qquad \text{for} \quad x = 2l-a_1t; \end{cases}$$

$$(2.24)\qquad \begin{cases} \sigma_8(x,t) = \sigma_{8p}(x)-\Delta\sigma_8(x,t), \\ \Delta\sigma_8 = 0 \qquad \text{for} \quad x = l+a_2(t-\tau). \end{cases}$$

Zone III, VI, and IX. We have

$$(2.25)\qquad v_3 = v_6 = v_9.$$

The equation of motion is

$$(2.26)\qquad [\varrho_1 a_1 l+\varrho_2 a_2(t-\tau_1)]\frac{dv_3}{dt} = P(t)+\sigma_{9p},$$

with the initial condition

$$(2.27)\qquad v_3 = v_2 \qquad \text{for} \qquad t = t^*,$$

$$(2.28)\qquad \varepsilon_3 = \varepsilon_{1p}, \qquad \varepsilon_6 = \varepsilon_{5p}, \qquad \varepsilon_9 = \varepsilon_{8p}.$$

For the remaining zone, σ_3 and σ will be computed according to formulae analogous to those for Zone II, except that for the determination of $\psi(t)$ we use the condition

$$(2.29)\qquad \begin{cases} \Delta\sigma_3 = \Delta\sigma_6 = 0 \qquad \text{for} \quad x = 2l-a_1t, \\ \Delta\sigma_9 = 0 \qquad \text{for} \quad x = l+a_2(t-\tau). \end{cases}$$

Zone IV, VII and X. We have, as before,

$$(2.30)\qquad v_4 = v_7 = v_{10},$$

$$(2.31)\qquad [\varrho_1 a_1 l+\varrho_2 a_2(t-\tau_1)]\frac{dv_4}{dt} = \sigma_{10p},$$

with the initial condition

$$(2.32)\qquad v_4(\tau) = v_3(\tau).$$

Next,

$$(2.33)\qquad \varepsilon_4 = \varepsilon_{1p}, \qquad \varepsilon_7 = \varepsilon_{5p}, \qquad \varepsilon_{10} = \varepsilon_{8p};$$

σ will be computed as before, with analogous conditions for $\Delta\sigma$, from which $\psi(t)$ is calculated. If we had $\tau_1 > \tau$ or $t^* > \tau$, the configuration of the zones would be different; the principle of constructing these solutions would, however, remain unchanged.

Let us consider now the second case, that is

Case 2. $|\sigma_{s1}| < |\sigma_{s2}|$. Similarly to the first case various solution variants and various zone configurations will be obtained depending on the assumed parameters τ, τ_1, l, σ_{s1}, σ_{s2} and P_0. We shall not, in view of limitations of space, discuss all the possible variants. We shall confine ourselves to a single case of essential

importance. The principle of constructing the solution will not be changed for another set of parameters, only different qualitative and quantitative conclusions will be obtained. To fix the attention, the set of parameters will be assumed as shown in Fig. 6.

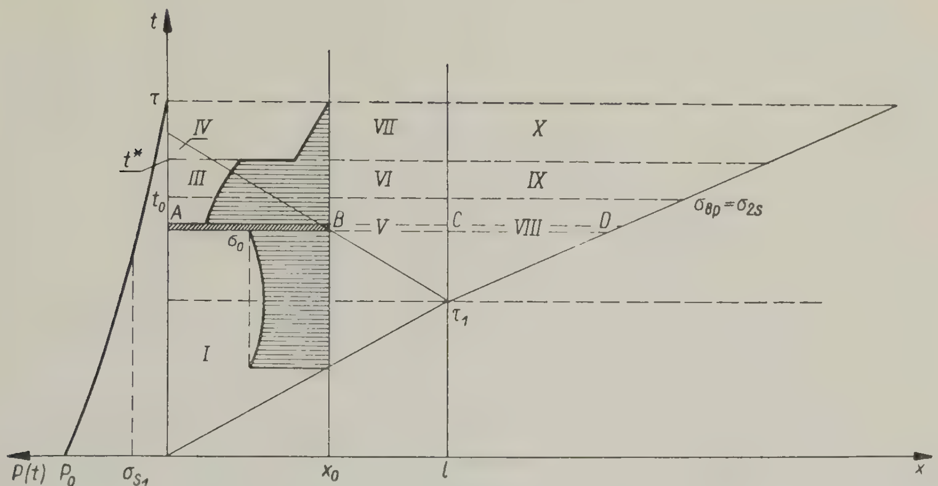

Fig. 6

The following solutions are obtained for each particular zone.

Zone I. The Eqs. (2.1)-(2.5) preserve their form except for $H(t)$ now replaced by

$$H(t) = \int_0^t [P(t_1)+\sigma_{1s}]dt_1, \tag{2.34}$$

and for the computation of σ the value σ_{s1} should be added to the right-hand members of the Eqs. (2.3) and (2.4).

Zone II. The equation of motion in Zone II is

$$\varrho_1(2l-a_1t)\frac{dv_2}{dt} = P(t)+\sigma_{1p}, \tag{2.35}$$

with the initial condition:

$$v_2(\tau_1) = \frac{1}{\varrho_1 a_1 \tau_1} H(\tau_1) = v_1(\tau_1). \tag{2.36}$$

Next,

$$\varepsilon_2 = \varepsilon_{1p}, \tag{2.37}$$

where, in view of a different definition of $H(t)$, ε_{1p} and σ_{1p} have sense other than in the case $\sigma_{1s} = \sigma_{2s} = 0$.

To calculate $\sigma_2(x, t)$, equations are available analogous to those for the previous case

(2.38) $$\sigma_2(x, t) = \sigma_{1p}(x) - \Delta\sigma_2(x, t),$$

(2.39) $$\Delta\sigma_2 = \int_0^x \left(\frac{\partial \sigma_{1p}}{\partial x} - \varrho_1 \frac{\partial v_2}{\partial t}\right) dx + P(t) - P_0.$$

Zone V and VIII. We have

(2.40) $$v_5 = v_8,$$

(2.41) $$\sigma_0 = -\varrho_1 a_1 (v_2 - v_5).$$

The equations of motion are:

(2.42) $$\begin{cases} \varrho_1(a_1 t - l)\dfrac{dv_5}{dt} = -(\sigma_0 + \sigma_{1p}) + \sigma_1 = \sigma_l + \varrho_1 a_1 (v_2 - v_5) - a_{1p}, \\ \varrho_2[a_2(t - \tau_1)]\dfrac{dv_8}{dt} = -\sigma_l + \sigma_{8p} = \sigma_l - \varrho_2 \sigma_2 v_8 + \sigma_{s2}, \qquad \sigma_{s2} < 0, \end{cases}$$

or, eliminating σ_l and making use of (2.40)

(2.43) $$[(\varrho_1 a_1 + \varrho_2 a_2)t - (\varrho_1 l + \varrho_2 a_2 \tau_1)]\frac{dv_5}{dt} + (\varrho_1 a_1 + \varrho_2 a_2) v_5 = -(\sigma_{1p} - \sigma_{s2}) + \varrho_1 a_1 v_2,$$

with the initial condition

(2.44) $$v_5(\tau_1) = 2v_1(\tau)\frac{\varrho_1 a_1}{\varrho_1 a_1 + \varrho_2 a_2} + \frac{\sigma_{s2} - \sigma_{s1}}{\varrho_1 a_1 + \varrho_2 a_2}.$$

In addition

(2.45) $$\varepsilon_5 = \frac{\sigma_8 - \sigma_{s1}}{\varrho_1 a_1^2}, \qquad \varepsilon_8 = \varepsilon_{8p}$$

and

(2.46) $$\begin{cases} \sigma_5(x, t) = \sigma_{1p}(x) + \sigma_0(x) - \Delta\sigma_5(x, t), \\ \sigma_8(x, t) = \sigma_8(x) - \Delta\sigma_8(x, t), \end{cases}$$

with the conditions for the determination of $\psi(t)$, analogous to (2.23) and (2.24).

Zone III, VI, IX. In the case of $t > t^\circ$, where t° — the time for which $\sigma_{8p} = \sigma_{s2}$, we have:

(2.47) $$v_6 = v_9 = 0,$$

(2.48) $$\varepsilon_3 = \varepsilon_{1p}, \quad \varepsilon_6 = \varepsilon_{5p}, \quad \varepsilon_9 = \varepsilon_{8p}, \quad (\varepsilon_9 = 0 \quad \text{for} \quad t > t_0).$$

v_3 will be calculated from an equation identical with (2.35), and the stresses according to equations analogous to those for Zone II.

Zones IV, VII and X. For $t > t^*$, where t^* is the time for which $\sigma_0 = 0$, we have

(2.49) $$v_4 = v_7 = v_{10} = 0,$$

(2.50) $$\varepsilon_4 = \varepsilon_{1p}, \quad \varepsilon_7 = \varepsilon_{5p}, \quad \varepsilon_{10} = \varepsilon_{8p}, \quad [\varepsilon_{10} = 0 \quad \text{for} \quad t > t_0],$$

(2.51) $$\sigma_4 = \sigma_7 = \sigma_{10} = -P(t).$$

For $t > \tau$, we have $\sigma = v = 0$, and permanent deformations take place as in Zones IV, VII and X.

Case 3. $|\sigma_{s1}| \geqslant |\sigma_{s2}|$. The parameters are assumed as shown in Fig. 7.

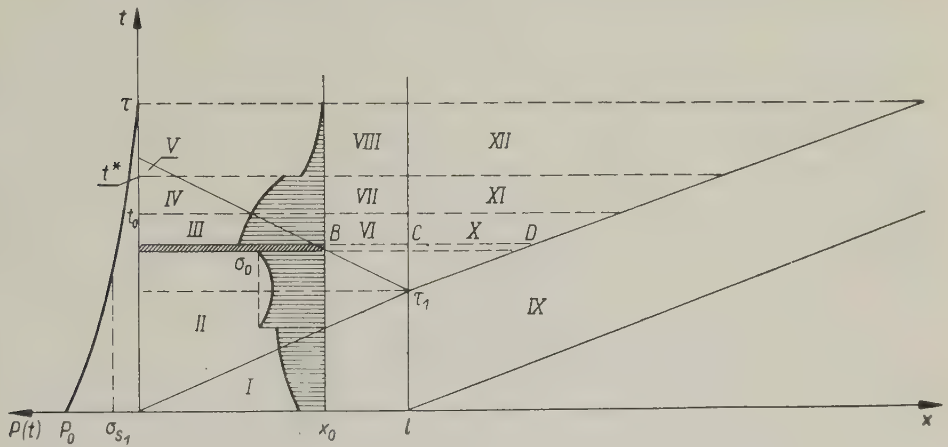

Fig. 7

In Fig. 7, it has been assumed that the pressure increases in the interval l, and decreases starting from τ_1. The case where for $\tau_0 < \tau_1$ the pressure reaches its maximum at the point l can occur only if $P(t)$ decreases in such a way that the plastic wave in Zone II ends before $x = l$ is reached. Then, in the medium, a wave of weak discontinuity appears (Fig. 8) [1].

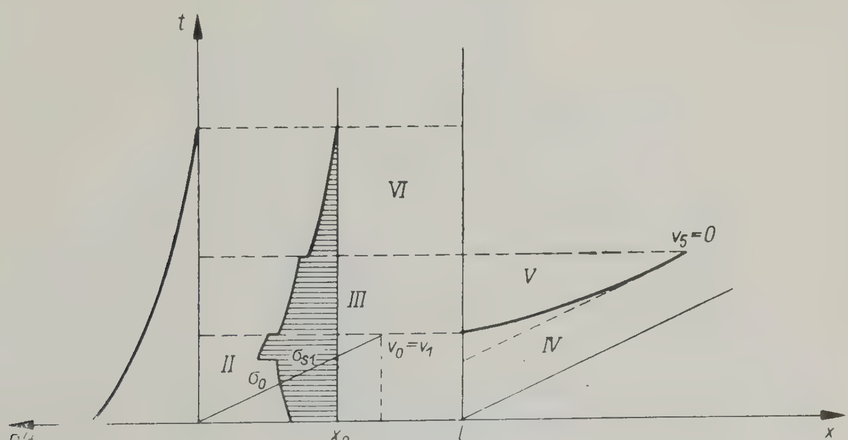

Fig. 8

This case is somewhat trivial and will not be discussed here. The possibility of appearance of such a maximum for $\tau_0 > \tau_1$ will not be discussed either, in view of the confined scope of the present paper.

Let us describe the case of Fig. 7, where σ_{max} takes place for τ_1. In the discussion of the equations for each particular zone, the equations will be written for v only. For, having v, we are able to determine σ and ε according to the above prescriptions.

Zone I. The equation of motion has the form:

$$\varrho_1(l-a_1t)\frac{dv_1}{dt} = -\sigma_{s1}+\sigma_l = -\sigma_{s1}+\sigma_{s2}-\varrho_2a_2v_1, \tag{2.52}$$

with the initial condition

$$v_1(0) = 0. \tag{2.53}$$

Zone IX.

$$v(x, t) = v_1\left(l-\frac{x-l}{a_2}\right). \tag{2.54}$$

Zone II. The equation of motion has the form;

$$\varrho_1a_1t\frac{dv_2}{dt} = P(t)+\sigma_{2p} = P(t)+\sigma_{s1}-\varrho_1a_1(v_2-v_1), \tag{2.55}$$

with the initial condition

$$v_2(0) = \frac{P_0+\sigma_{s1}}{\varrho_1a_1}. \tag{2.56}$$

Zone III. The equation of motion has the form:

$$\varrho_1(2l-a_1t)\frac{dv_3}{dt} = P(t)+\sigma_{2p}(2l-a_1t) \ . \tag{2.57}$$

with the initial condition

$$v_3(\tau_1) = v_2(\tau_1). \tag{2.58}$$

Zone VI and X. In these zones, we have:

$$v_6 = v_{10}, \tag{2.59}$$

$$\sigma_0 = -\varrho_1a_1(v_3-v_6), \tag{2.60}$$

$$\left\{\begin{aligned} &\varrho_1(a_1t-l)\frac{dv_6}{dt} = -(\sigma_0+\sigma_{1p})+\sigma_l = \sigma_l+\varrho_1a_1(v_3-v_6)-\sigma_{2p},\\ &\varrho_2a_2(t-\tau_1)\frac{dv_{10}}{dt} = -\sigma_l+\sigma_{10p} = -\sigma_l+\sigma_{s2}-\varrho_2a_2(v_{10}-v_9)_s \end{aligned}\right. \tag{2.61}$$

or, eliminating σ_l,

$$\begin{aligned}[(\varrho_1a_1+\varrho_2a_2)t-(\varrho_1l+\varrho_2a_2\tau_1)]\frac{dv_6}{dt}+(\varrho_1a_1+\varrho_2a_2)v_6 = \\ = -(\sigma_{2p}-\sigma_{s2})+\varrho_1a_1v_3+\varrho_2a_2v_9,\end{aligned} \tag{2.62}$$

with the initial condition

$$v_6(\tau_1) = v_2(\tau_1)\frac{\varrho_1a_1}{\varrho_1a_1+\varrho_2a_2}+\frac{\sigma_{s2}-\sigma_{s1}+\varrho_1a_1v_1+\varrho_2a_2v_9}{\varrho_1a_1+\varrho_2a_2}, \tag{2.63}$$

where

$$v_9 = v_9(\tau).$$

In the remaining zones we obtain, bearing in mind the changes in numeration of the zones, the same relations as in case 2.

In view of the limited scope of this paper, the discussion of other variants for various values of the parameters P_0, σ_{s1}, σ_{s2}, l etc. assumed will be omitted.

In spite of different qualitative and quantitative results for different values of these parameters, the general approach will remain unchanged.

The choice of the parameters for the above cases has been so made that the most essential cases are discussed, the ones which are trivial or similar to those discussed above being omitted.

3. The Unloading Wave in the Case of $\varrho_1 a_1 > \varrho_2 a_2$

Three cases should be considered, as in Sec. 2.

$$(1)\ \sigma_{s1} = \sigma_{s2} = 0, \qquad (2)\ |\sigma_{s1}| < |\sigma_{s2}|, \qquad (3)\ |\sigma_{s1}| > |\sigma_{s2}|.$$

Here also a considerable number of variants is obtained, which cannot be discussed within the framework of a brief paper. However, since the essential features of

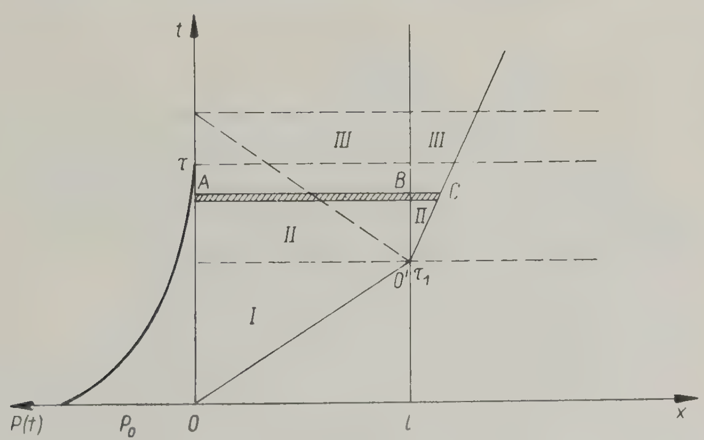

Fig. 9

the method remain unchanged, as compared with Sec. 2, the discussion of the cases (2) and (3) will be omitted, case (1) being the only case discussed as an example

$$\text{Case } 1: \sigma_{s1} = \sigma_{s2} = 0.$$

Let us assume that the parameters in the phase plane are as shown in Fig. 9 where $a_2 < a_1$ has been assumed, which corresponds to the condition

$$\varrho_1 a_1 > \varrho_2 a_2 \qquad \text{for} \qquad \varrho_1 = \varrho_2 = \varrho.$$

This case is much simpler than the analogous case of Sec. 2. Let us consider now the equations and the solutions for each particular zone.

Zone I. The equations of Sec. 2, for the case of $\sigma_{s1} = \sigma_{s2} = 0$, preserve their validity.

Zone II. The equation of motion for Zone II is the equation for the unloading zone, and the stress at the point O' will undergo a jump-like decrease. This equation has the form:

$$[\varrho_1 l + \varrho_2 a_2 (t - \tau_1)] \frac{dv_2}{dt} = P(t) - \varrho_2 a_2 v_2, \tag{3.1}$$

with the initial condition:

$$v_2(\tau_1) = v_1(\tau_1). \tag{3.2}$$

The distributions σ_2 and ε_2 will be calculated similarly to Sec. 2.

Zone III. The equation of motion has the form

$$[\varrho_1 l + \varrho_2 a_2 (t - \tau_1)] \frac{dv_3}{dt} + \varrho_2 a_2 v_2 = 0, \tag{3.3}$$

with the initial condition

$$v_3(\tau) = v_2(\tau). \tag{3.4}$$

σ_3 and ε_3 will be computed in the same manner as in the foregoing section.

4. Example

Two examples will be considered. In view of the great number and variety of cases, and the limitet extent of this paper, let us consider only some examples for the case 1, that is for $\sigma_{s1} = \sigma_{s2}$ of the Sec. 2 and 3, in view of their simplicity and practical importance in soil mechanics where the simplified diagrams can be applied to approximate the real $(\sigma - \varepsilon)$-diagram (Fig. 4).

Example 1.

Let us make the following assumptions:

$$\begin{cases} \varrho_1 = \varrho_2 = \varrho, & a_1 < a_2, \\ P(t) = P_0\left(1 - \dfrac{t}{\tau}\right) & \text{for} \quad t < \tau, \\ P(t) = 0 & \text{for} \quad t > \tau. \end{cases} \tag{4.1}$$

Then, in agreement with the considerations of Sec. 2, the solutions for each particular zone are as follows.

Zone I.

$$v_1 = v_{1p} = \frac{P_0}{\varrho a_1}\left(1 - \frac{t}{2\tau}\right), \tag{4.2}$$

$$\sigma_1 = -P_0\left(1 + \frac{x}{2a_1\tau} - \frac{t}{\tau}\right), \tag{4.3}$$

$$\sigma_{1p} = -P_0\left(1-\frac{x}{2a_1\tau}\right), \tag{4.4}$$

$$\varepsilon_1 = \varepsilon_{1p} = -\frac{P_0}{E_1^*}\left(1-\frac{x}{2a_1\tau}\right) = -\frac{P_0}{\varrho a_1^2}\left(1-\frac{x}{2a_1\tau}\right). \tag{4.5}$$

Zone II. The equation of motion (2.6) becomes

$$\varrho(2l-a_1t)\frac{dv_2}{dt} = P_0\left(1-\frac{t}{\tau}\right) - P_0\left(1-\frac{2l-a_1t}{2a_1\tau}\right) = -P_0\left(\frac{3}{2}\frac{t}{\tau}-\frac{\tau_1}{\tau}\right) \tag{4.6}$$

with the initial condition, (2.7):

$$v_2(\tau_1) = v_1(\tau_1) = \frac{P_0}{\varrho a_1}\left(1-\frac{\tau_1}{2\tau}\right). \tag{4.7}$$

On integrating, we obtain:

$$v_2 = \frac{2P_0}{\varrho a_1}\left[\frac{\tau_1}{\tau}\ln\left(2-\frac{t}{\tau_1}\right)+\frac{3}{4}\frac{t}{\tau}+\frac{1}{2}-\frac{\tau_1}{\tau}\right]. \tag{4.8}$$

Next,

$$\varepsilon_2 = -\frac{P_0}{\varrho a_1^2}\left(1-\frac{x}{2a_1\tau}\right). \tag{4.9}$$

and, on the basis of the Eqs. (2.9) — (2.13):

$$\sigma_2(x,t) = -P_0\left(1+\frac{x}{2a_1\tau}-\frac{t}{\tau}+\frac{2x}{2l-a_1t}\,\frac{t-\tau_1}{\tau}\right). \tag{4.10}$$

Zones V and VIII. We have $v_5 = v_8$ and the equation of motion for both zones (2.20) is

$$\varrho[(a_1+a_2)t-(l+a_2\tau_1)]\frac{dv_5}{dt}+\varrho(a_1+a_2)v_5 = -\sigma_{1p}+\varrho a_1 v_2 = \tag{4.11}$$

$$= P_0\left(1-\frac{2l-a_1t}{2a_1\tau}\right)+2P_0\left[\frac{\tau_1}{\tau}\ln\left(2-\frac{t}{\tau_1}\right)+\frac{3}{4}\frac{t}{\tau}+\frac{1}{2}-\frac{\tau_1}{\tau}\right] =$$

$$= 2P_0\left[\frac{\tau_1}{\tau}\ln\left(2-\frac{t}{\tau_1}\right)+\frac{t}{\tau}+1-\frac{3}{2}\frac{\tau_1}{\tau}\right],$$

or, after transformation,

$$(t-\tau_1)\frac{dv_5}{dt}+v_5 = \frac{2P_0}{\varrho(a_1+a_2)}\left[\frac{\tau_1}{\tau}\ln\left(2-\frac{t}{\tau_1}\right)+\frac{t}{\tau}+1-\frac{3}{2}\frac{\tau_1}{\tau}\right] = \Phi(t). \tag{4.12}$$

The solution of the Eq. (4.12) may be represented in the form:

$$v_5(t) = \frac{1}{t-\tau_1}\left[\int_{\tau_1}^{t}\Phi(t_1)dt_1+C\right]. \tag{4.13}$$

From the initial condition (2.21), it follows that $C = 0$ because for $t \to \tau_1$, in the expression (4.13), we obtain, for $C = 0$,

$$v_5(\tau_1) = \frac{2P_0}{\varrho(a_1+a_2)}\left(1-\frac{\tau_1}{2\tau}\right), \tag{4.14}$$

which coincides with the initial condition (2.21). Writing out (4.13) we obtain after some manipulation:

$$(4.15)\qquad v_8(t) = v_5(t) = \frac{2P_0}{\varrho(a_1+a_2)(t-\tau_1)}\Bigg[-\frac{\tau_1^2}{2\tau}+\frac{t^2}{2\tau}+ +\left(1-\frac{5}{2}\frac{\tau_1}{\tau}\right)(t-\tau_1)-\frac{\tau_1^2}{\tau}(2\tau_1-t)\ln\frac{2\tau_1-t}{\tau_1}\Bigg].$$

Next,

$$(4.16)\qquad \varepsilon_5 = -\frac{1}{\varrho a_1^2}\Bigg\{P_0\left(1-\frac{2l-a_1t}{2a_1t}\right)+2P_0\left[\frac{\tau_1}{\tau}\ln\left(2-\frac{t}{\tau_1}\right)+\frac{3}{4}\frac{t}{\tau}+\frac{1}{2}-\frac{\tau_1}{\tau}\right]- -\frac{2P_0}{\varrho(a_1+a_2)(t-\tau_1)}\left[\frac{t^2}{2\tau}+\left(1-\frac{5}{2}\frac{\tau_1}{\tau}\right)(t-\tau_1)-\frac{\tau_1}{\tau}(2\tau_1-t)\ln\frac{2\tau_1-t}{\tau_1}\right]\Bigg\},$$

$$(4.17)\qquad \varepsilon_8 = -\frac{v_8\left(t\quad\dfrac{x-l}{a_2}+\tau_1\right)}{a_2}$$

Similarly, by virtue of the equations of the type (2.9)-(2.13) and making use of (2.23) and (2.24), we obtain:

$$(4.18)\qquad \sigma_5(x,t) = -P_0\Bigg\{1-\frac{2l-a_1t}{2a_1\tau}+2\left(\frac{\tau_1}{\tau}\ln\frac{2l-a_1t}{a_1\tau_1}+\frac{3}{4}\frac{t}{\tau}+\frac{1}{2}-\frac{\tau_1}{\tau}\right)- -\frac{2a_1}{(a_1+a_2)(t-\tau_1)}\left[\frac{t^2-\tau_1^2}{2\tau}+\left(1-\frac{5}{2}\frac{\tau_1}{\tau}\right)(t-\tau_1)-\frac{\tau_1}{\tau}(2\tau_1-t)\ln\frac{2l-a_1t}{a_1\tau_1}\right]\Bigg\}- -\frac{2P_0}{a_1+a_2}\left\{\frac{1}{t-\tau_1}\left[\frac{t+\tau_1}{2\tau}+\ln\frac{2\tau_1-t}{\tau_1}\left(\frac{\tau_1^2}{\tau(t-\tau_1)}\right)\right]\right\}(x-2l+a_1t).$$

$$(4.19)\qquad \sigma_8(x,t) = -\frac{2P_0}{(a_1+a_2)(t-\tau_1)}\Bigg\{\Bigg[\frac{t^2-\tau_1^2}{2\tau}+\left(1-\frac{5}{2}\frac{\tau_1}{\tau}\right)(t-\tau_1)- -\frac{\tau_1}{\tau}(2\tau_1-t)\ln\frac{2\tau_1-t}{\tau_1}\Bigg]a_2+\left[\frac{t+\tau_1}{2\tau}+\ln\frac{2\tau_1-t}{\tau_1}\frac{\tau_1^2}{\tau(t-\tau_1)}\right]\Big[x-l-a_2(t-\tau_1)\Big]\Bigg\}.$$

For $a_2 = \infty$, that is for the case of incidence and reflection of the unloading wave in the first layer from a perfectly rigid wall, we obtain

$$(4.20)\qquad \sigma_5 = -2P_0\left[1+\frac{t}{\tau}-\frac{3}{2}\frac{\tau_1}{\tau}+\frac{\tau_1}{\tau}\ln\left(2-\frac{t}{\tau_1}\right)\right].$$

This formula coincides with the formula obtained in Ref. [2] for the discussion of the problem of reflection of the unloading wave from a rigid wall.

Zones III, VI and IX. In what follows, we shall give only the computation of v. Having v, the quantities ε and σ can be computed as before. Since $v_3 = v_6 = v_9$, in agreement with (2.25), therefore the equation of motion (2.26) has the form

$$(4.21)\qquad \varrho[l+a_2(t-\tau_1)]\frac{dv_3}{dt} = P_0\left(1-\frac{t}{\tau}\right)+\varrho a_2 v_3,$$

with the initial condition

$$v_3(t^*) = v_2(t^*). \tag{4.22}$$

Integrating this, and making use of the initial condition (4.22), we obtain, after rearrangement:

$$v_3(t) = \frac{P_0}{\varrho[l+a_2(t-\tau_1)]}\left\{t\left(1-\frac{t}{2\tau}\right)+\frac{2}{a_1}\left[\frac{\tau_1}{\tau}\ln\left(2-\frac{t^*}{\tau}\right)+ \right.\right.$$
$$\left.\left. +\frac{3}{4}\frac{t^*}{\tau}+\frac{1}{2}-\frac{\tau_1}{\tau}\right][l+a_2(t^*-\tau_1)]-t^*\left(1-\frac{t^*}{2\tau}\right)\right\}. \tag{4.23}$$

The value of t^* will be computed from:

$$v_2 = v_5. \tag{4.24}$$

It can be computed by assuming a numerical value of τ_1/τ (in view of the character of the Eq. (4.24), usually transcendental). It has been assumed that $\tau_1 < t^* < \tau$ and $\tau_1 > \tau/2$, in agreement with Fig. 5.

Zones IV, VIII and X. In agreement with (2.30) and (2.31), we have

$$v_4 = v_7 = v_{10}$$

and

$$\varrho[l+a_2(t-\tau_1)]\frac{dv_4}{dt}+\varrho a_2 v_4 = 0, \tag{4.25}$$

with the initial condition

$$v_4(\tau) = v_3(\tau). \tag{4.26}$$

The Eq. (4.25) is identical with (4.21) if it is assumed that $P(t) = 0$. Integrating, this, we obtain:

$$v_4(t) = \frac{P_0}{[l+a_2(t-\tau_1)]\varrho}\left\{\frac{\tau^2}{2}+\frac{2}{a_1}\left[\frac{\tau_1}{\tau}\ln\left(2-\frac{t^*}{\tau}\right)+ \right.\right.$$
$$\left.\left. +\frac{3}{4}\frac{t^*}{\tau}+\frac{1}{2}-\frac{\tau_1}{\tau}\right][l+a_2(t^*-\tau_1)]-t^*\left(1-\frac{t^*}{2\tau}\right)\right\}. \tag{4.27}$$

The above remark on t^* remains valid.

Thus the full solution is obtained in a closed form for the case considered.

Example 2. The data are the same as in the Example 1, except that $a_1 > a_2$ is assumed (Fig. 9). Then, in agreement with the results of Sec. 3, we obtain:

Zone I. The same solution is obtained as in Zone I of the first example. It is given by the Eqs. (4.2)-(4.5).

Zone II. The equation of motion (3.1) has the form

$$\frac{dv_2}{dt}+\frac{a_2}{l+a_2(t-\tau_1)}v_2 = \frac{P_0}{\varrho}\frac{1-\dfrac{t}{\tau}}{l+a_2(t-\tau_1)}, \tag{4.28}$$

with the initial condition (3.2)

$$v_2(\tau_1) = v_1(\tau_1) = \frac{P_0}{\varrho a_1}\left(1-\frac{\tau_1}{2\tau}\right). \tag{4.29}$$

On integrating, we obtain:

$$v_2(t) = \frac{P_0}{\varrho}\,\frac{t\left(1-\dfrac{t}{2\tau}\right)}{l+a_2(t-\tau_1)}, \tag{4.30}$$

$$\begin{cases} \varepsilon_2 = \varepsilon_{1p} \text{ for } \quad t<\tau_1 \quad (\varepsilon_{1p} \text{ is given by the Eq. (4.5)}), \\ \varepsilon_2 = \left[-\dfrac{v_2}{a_2}\right]_{t=\frac{x-l}{a_2}+\tau} \qquad \text{for} \quad t>\tau_1, \end{cases} \tag{4.31}$$

$$\sigma_2(x,t) = \sigma_{2p}(x) - \Delta\sigma_2(x,t), \tag{4.32}$$

where

$$\Delta\sigma_2(x,t) = \int\limits_{l+a_2(t-\tau_1)}^{x} \left(\frac{\partial\sigma_{2p}}{\partial x} - \varrho\frac{\partial v_2}{\partial t}\right)dx; \tag{4.33}$$

hence, after performing the calculations necessary:

$$\sigma_2(x,t) = P_0\left\{-a_2\frac{t\left(1-\dfrac{t}{2\tau}\right)}{l+a_2(t-\tau_1)} + \right. \tag{4.34}$$

$$\left. +\frac{1}{2\tau}\,\frac{2(\tau-t)[l+a_2(t-\tau_1)]-(2\tau-t)ta_2}{[l+a_2(t-\tau_1)]^2}[x-l-a_2(t-\tau_1)]\right\},$$

$$\begin{cases} \sigma_1 = -P_0\left(1-\dfrac{\tau_1}{2\tau}\right), \\ \sigma_2 = -P_0\dfrac{a_2}{a_1}\left(1-\dfrac{\tau_1}{2\tau}\right) \qquad \text{for} \quad t=\tau_1, \quad x=l. \end{cases} \tag{4.35}$$

The stress jump is:

$$\sigma_1-\sigma_2 = -P_0\left(1-\frac{\tau_1}{2\tau}\right)\frac{a_1-a_2}{a_2}. \tag{4.36}$$

If $a_2 = 0$, we have $\sigma_2 = 0$, and the end is free. (The case of $a_2 = \infty$ was discussed in the foregoing example. Now, in view of $a_1 > a_2$ this limit case cannot be investigated). If $a_1 = a_2$, we have $\sigma_1 = \sigma_2$.

Zone III. From the Eq. (4.28), for $P(t) = 0$ and the initial condition (3.4), we obtain:

$$v_3(t) = \frac{P_0\tau}{2\varrho}\,\frac{1}{l+a_2(t-\tau_1)}. \tag{4.37}$$

ε_3 is expressed by equations analogous to (4.31)

$$\sigma_3(x,t) = a_2P_0\left\{-\frac{\tau}{2[l+a_2(t-\tau_1)]} - \frac{\tau}{2}\,\frac{x-l-a_2(t-\tau_1)}{[l+a_2(t-\tau_1)]^2}\right\}. \tag{4.38}$$

5. Conclusion

The above solutions are obtained, as in [2], in a closed form owing to the assumption of rigid unloading characteristic, which leads, as indicated above, to certain qualitative changes in the process, but presents a relatively good quantitative approach to the reality. The combination of a great variety of cases with space restrictions placed on the present paper necessitates on discussing, only the most important cases. The rest may be solved by using the same methods. The assumption of a modified rigid unloading characteristic makes it possible to obtain the solution also, for the three dimensional states of stress for spherical and cylindrical waves, which is very simple in relation to the known numerical solutions. This will be the object of separate paper.

References

[1] S. Kaliski, J. Osiecki, *Unloading wave for a body with rigid unloading characteristic*, Proc. Vibr. Probl., 1, **1** (1959).

[2] S. Kaliski, J. Osiecki, *The problem of reflection by rigid or elastic wall of an unloading wave in a body with rigid unloading characteristic*, Proc. Vibr. Probl., 1, **1** (1959).

[3] E. H. Lee, H. Wolf, *Plastic wave propagation effects in high-speed testing*, J. Appl. Mech.. 4, **18** (1951).

[4] J. Osiecki, *Reflection of plane stress wave in a non-homogeneous medium*, Proc. Vibr. Probl., 2, **2** (1961).

Streszczenie

FALA ODCIĄŻENIA DLA CIAŁA O SZTYWNEJ CHARAKTERYSTYCE ODCIĄŻENIA W OŚRODKU WARSTWOWYM

W pracy podano ścisłe i zamknięte rozwiązania dla rozprzestrzeniania się plastycznej fali odciążenia w ośrodku warstwowym dla ciała o sztywnej charakterystyce odciążenia, przy granicy sprężystości równej zeru lub różnej od zera. Gdy $\sigma_s \neq 0$, to zakłada się również sztywne obciążenie do granicy sprężystości.

W przypadku gdy $a_2 = \infty$ otrzymuje się rozwiązania otrzymane w pracy [2] dla odbicia się fali od sztywnej ściany przy $\sigma_1 = 0$.

W przypadku ośrodka warstwowego i $\sigma_1 \neq 0$ zjawisko rozprzestrzeniania się fal ulega komplikacji. Ze względu na liczne możliwe warianty przypadków w zależności od przyjętych parametrów przedyskutowano zasadnicze spośród nich.

Rozwiązania dla pozostałych przypadków buduje się analogicznie. Sztywna charakterystyka odciążenia daje pewne jakościowe zmiany w obrazie rozprzestrzenienia się fal, wyniki ilościowe otrzymuje się jednakże bliskie rzeczywistości. Poza tym odpada błąd związany z przybliżeniami matematycznymi, gdyż rozwiązania otrzymuje się w postaci zamkniętej.

Резюме

ВОЛНА РАЗГРУЗКИ ДЛЯ ТЕЛА, ОБЛАДАЮЩЕГО ЖЕСТКОЙ ХАРАКТЕРИСТИКОЙ РАЗГРУЗКИ В СЛОИСТОЙ СРЕДЕ

Дается точное и замкнутое решение для распространения пластической волны разгрузки в слоистой среде для тела, обладающего жесткой характеристикой разгрузки, при пределе упругости равному нулю или не равному нулю. Когда $\sigma_s \neq 0$ предполагается отсутствие упругих деформаций.

В случае если $a_2 = \infty$, получается решение, полученное уже в работе [2] для отражения волны от жесткой стены при $\sigma_s = 0$.

В случае слоистой среды и $\sigma_s \neq 0$ распространение волны усложняется. Принимая во внимание многочисленные, возмож ые варианты случаев, в зависимости от принятых параметров, проводится дискуссия принципиальных вариантов.

Решение для остальных случаев составляется аналогично. Жесткая характеристика разгрузки дает некоторые качестьенные изменения в картине распространения волн, однако количественные результаты близки действительности. Кроме того, можно избежать ошибки, связанной с математическими приближениями так как решения получаются в замкнутом виде.

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